DYNAMICS OF A DISCRETE-TIME MODEL OF AN “IDEAL-STORAGE” SYSTEM DESCRIBING HETERO-CATALYTIC PROCESSES ON METAL SURFACES

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In this paper, we analyze a rather simple system in which some substance is being stored, released, and replenished simultaneously in some interdependent way. We investigate the dynamic behavior of such a system, using a two-dimensional map-based discrete-time model, and derive an integrated dynamical scene for this model. More specifically, we show the existence of an invariant curve induced by the well-known Neimark-Sacker bifurcation corresponding to the presence of a periodically oscillating behavior in this model.

Keywords: Map-based model, Periodic behavior, Neimark-Sacker bifurcation, heterogeneous catalytic reactions.

1. Introduction

Oscillations, in particular periodic oscillations, are omnipresent phenomena in both, nature and man-made systems. Examples abound: varieties of oscillations are experimentally observed in heterogeneous catalytic and electrochemical reactions; periodic and even chaotic oscillations are experimentally and numerically found in neuron activities; rhythms arise in genetic and metabolic networks as a result of various modes of cellular regulation [Hodgkin & Huxley, 1990; Llauw et al., 1996; Rocsoreanu et al., 2000; Rulkov, 2002; Goldbeter, 2002; Pomerening, 2005; Tsai, 2008; Baker et al., 2009; Lin & Chen, 2009]. In order to understand the mechanisms underlying these phenomena and to even make reliable predictions, researchers usually establish mathematical models involving either continuous or discrete dynamical systems. In this paper, we analyze a model that was designed to describe the oscillating behavior of a system in which some substance
is being stored, released, and replenished simultaneously in some interdependent way. More specifically, following [Dress et al., 1982], we consider the one-parameter system of maps \( \varphi_c \)

\[
\varphi_c : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_1 x_2 + c \\ f(x_1 x_2) \end{pmatrix}
\]

(1)

where \( c \) is a positive constant and \( f \) is a monotonically increasing function defined on \( \mathbb{R}^+ := [0, +\infty) \) whose range is contained in \((0, 1)\). We will view \( \varphi_c \) as a map from the product space \( \mathcal{X} := [0, +\infty) \times [0, 1] \subset \mathbb{R}^2 \) into itself, depending on a positive control parameter \( c \). Consequently, our discrete-time map-based model can be written in the form of

\[
\begin{align*}
x_1(n + 1) &= x_1(n)[1 - x_2(n)] + c, \\
x_2(n + 1) &= f(x_1(n)x_2(n)),
\end{align*}
\]

(2)

\( n = 0, 1, 2, \cdots \) — or, with \( x = (x_1, x_2)^\top \), in a more compact form \( x(n + 1) = \varphi_c(x(n)) \).

Model (2) is meant to describe the dynamical behavior of a system in which, at time step \( n \), the amount of \( x_1(n) \) “units of mass” of some substance \( X_1 \) has been stored. At the next time step, part of the stored amount is released — at a rate given by \( x_2(n) \) — and, simultaneously, it is continuously replenished by the constant amount \( c \). The first equation in Model (2) can, thus, be viewed as just a simple balance (or bookkeeping) equation: What is there at time \( n+1 \), is the sum of what was there at time \( n \) plus the constant influx \( c \) minus all that has been released which must be a certain fraction \( x_2(n) \) of the amount \( x_1(n) \) that was there before. Thus, this equation sort of just defines the fraction \( x_2(n) = (x_1(n+1) - x_1(n) - c)/x_1(n) \) of the amount \( x_1(n) \) of substance that is released in-between the time points \( n \) and \( n+1 \). The second equation, then, presents the model assumption: It claims that this fraction \( x_2 \) depends, in a monotonously increasing way, on the total amount \( x_1 x_2 \) of substance that has been released during the last time step. It follows from the presumed monotonicity of the function \( f \) that this rate becomes larger in the current time step in case that total amount of released substance increased compared to the previous time step — and becomes smaller if this amount decreased. In what follows, we will verify the global boundedness of the orbit \( \mathcal{O}^+ = \mathcal{O}^+(x(0)) := \{x(n)\}_{n=0}^\infty \) generated by our Model (2), analyse the local stability of its unique fixed point, and investigate the existence and stability of the periodic oscillations induced, under appropriate conditions, by a bifurcation of the well-known Neimark-Sacker type around that fixed point.

2. Boundedness and Local Stability

It follows from the particular form of the map \( \varphi_c \) that the open subset \( R := (0, \infty) \times (0, 1) \) of \( \mathbb{R}^2 \) is invariant under this map. This implies that the orbit \( \mathcal{O}^+ \) of model (2) never leaves \( R \) as long as the initial value is arbitrarily taken within \( R \), that is, whenever \( 0 < x_1(0) \) and \( 0 < x_2(0) < 1 \) holds. Thus, the second component \( x_2(n) \) of the orbit \( \mathcal{O}^+ \) is always bounded from above by 1 and from below by 0 — or, after the first iteration, even by \( f(0) \) — and we only need to investigate the boundedness of the first component \( x_1(n) \). On the one hand, it is obvious that this component is bounded from below — again, at least, after the first iteration — by the constant \( c \). On the other hand, induction easily yields that

\[
x_1(n) \leq x_1(0) \left[ 1 - f(0) \right] + \frac{c - c \left[ 1 - f(0) \right]}{f(0)}
\]

\[
= \frac{c}{f(0)} + \left[ x_1(0) - \frac{c}{f(0)} \right] \left[ 1 - f(0) \right] \quad \text{if } x_1(0) \geq \frac{c}{f(0)} \text{ holds,}
\]

\[
\leq \max(x_1(0), \frac{c}{f(0)}) \quad \text{otherwise,}
\]

\[
\lim_{n \to \infty} x_1(n) = \frac{c}{f(0)}.
\]
holds for all $n \geq 0$: Indeed, this inequality clearly holds, as an equality, for $n = 0$. And, if it holds for some $n$, this will imply that also
\[
x_{1}(n+1) = x_{1}(n)[1 - x_{2}(n)] + c \\
\leq \left( x_{1}(0)[1 - f(0)] + \frac{c - c[1 - f(0)]^{n}}{f(0)} \right) [1 - f(0)] + c \\
= x_{1}(0)[1 - f(0)]^{n+1} + \frac{c[1 - f(0)] - c[1 - f(0)]^{n+1}}{f(0)} + c \\
= x_{1}(0)[1 - f(0)]^{n+1} + \frac{c - c[1 - f(0)]^{n+1}}{f(0)} \\
= \frac{c}{f(0)} + \left[ x_{1}(0) - \frac{c}{f(0)} \right] [1 - f(0)]^{n+1}
\]
holds. Clearly, this implies that

(i) $x_{1}(n+1) < \frac{c}{f(0)} + \epsilon$ must hold, for every positive $\epsilon$, for all sufficiently large integers $n$ (in view of $\lim_{n \to \infty}[1 - f(0)]^{n+1} = 0$),

(ii) $x_{1}(n) \leq \frac{c}{f(0)}$ must hold for all $n \geq 0$ in case $x_{1}(0) \leq \frac{c}{f(0)}$ holds, implying also that $x_{1}(n) \leq \frac{c}{f(0)}$ holds for all $n \geq n_{0}$ provided $x_{1}(n_{0}) \leq \frac{c}{f(0)}$ holds for some natural number $n_{0} \in \mathbb{N}$, and

(iii) it implies also that such a number $n_{0}$ necessarily exists in case $f(c) > f(0)$ holds: Indeed, assuming that, to the contrary, $x_{1}(n) > \frac{c}{f(0)}$ holds for all $n \geq 0$, we get
\[
x_{2}(n+1) = f(x_{1}(n)x_{2}(n)) \geq f \left( \frac{c}{f(0)}x_{2}(n) \right) \geq f \left( \frac{c}{f(0)f(0)} \right) = f(c) > f(0),
\]
for all $n \geq 1$ and, therefore, assuming that
\[
x_{1}(n_{1}) \leq \frac{c}{f(0)} + \epsilon
\]
holds, for some natural number $n_{1} \in \mathbb{N}$, for $\epsilon := c - \frac{c}{f(0)} - \frac{c}{f(0)[1 - f(c)]}$, we get
\[
x_{1}(n_{1}+1) = x_{1}(n_{1})[1 - x_{2}(n_{1})] + c \\
\leq \left( \frac{c}{f(0)} + \epsilon \right)[1 - f(c)] + c \\
= \frac{c}{f(0)} + \frac{c - f(0) - f(c)}{f(0)} + \epsilon[1 - f(c)] \\
= \frac{c}{f(0)} + \frac{c - f(0) - f(c)}{f(0)} + \epsilon[1 - f(c)] \\
= \frac{c}{f(0)}
\]
a contradiction,

(iv) while in case $f$ is, say, constant, we have
\[
x_{1}(n+1) - \frac{c}{f(0)} = x_{1}(n)[1 - f(0)] + c - \frac{c}{f(0)} = \left[ x_{1}(n) - \frac{c}{f(0)} \right] [1 - f(0)]
\]
for all $n \geq 1$ and, therefore,
\[
x_{1}(n) - \frac{c}{f(0)} = \left[ x_{1}(1) - \frac{c}{f(0)} \right] [1 - f(0)]^{n-1},
\]
that is, $x_{1}(n)$ converges monotonically towards $\frac{c}{f(0)}$: from above in case $x_{1}(1) \geq \frac{c}{f(0)}$, and from below in case $x_{1}(1) \leq \frac{c}{f(0)}$.

It is also clear that the bounded first component could be oscillating in a broad range if the value of $f(0)$ is sufficiently small. This will be further reinforced by numerical simulations in the following section. In summary, the above argument yields the following theorem.
Theorem 1. The orbit $O^+$ of Model (2), initiating from any point in $R$, is globally bounded. More accurately, one has

$$ 0 < x_1(n) \leq \max \left( x_1(0), \frac{c}{f(0)} \right) $$

and

$$ f(0) \leq x_2(n + 1) = f(x_1(n)x_2(n)) \leq f(x_1(n)) \leq f \left( \max(x_1(0), \frac{c}{f(0)}) \right) < 1, $$

for all $n = 1, 2, \ldots$, provided that $x_1(0) > 0$ and $x_2(0) \in (0, 1)$ holds, one has $0 < x_1(n) < \frac{c}{f(0)} + \epsilon$, for every positive $\epsilon$, for every sufficiently large $n$, and we have $0 < x_1(n) < \frac{c}{f(0)}$ for every sufficiently large $n$ provided $f(c) > f(0)$ holds. In particular, we have

$$ \emptyset \neq \lim_{n \to \infty} O^+ (x(0)) := \{ x : \forall \epsilon > 0, \forall n \in N, \exists n' > n, \| x - \varphi^n_c (x) \| < \epsilon \} \subseteq (0, \frac{c}{f(0)}] \times [f(0), f \left( \max(x_1(0), \frac{c}{f(0)}) \right) ], $$

for every orbit $O^+ (x(0)) = \{ \varphi^n_c (x) \}_{n=0}^{\infty}$.

It also follows from our equations that there is a unique fixed point $E = E(c)$ of Model (2), i.e., the point $E = \left( \frac{c}{f(c)}, f(c) \right)^T$. To investigate the local stability of this fixed point, we may assume that $f(x)$ is differentiable at $x = c$ (recall that, according to Lebesgue, every monotonically increasing function defined on some interval contained in $R$ is anyway differentiable almost everywhere) in which case we can consider the eigenvalues of the Jacobian matrix of the map $\varphi_c$ at $E$, i.e., the matrix

$$ J_1 := D \varphi_c (x) \bigg|_{x=E} = \begin{pmatrix} 1 - f(c) & \frac{c}{f(c)} & -\frac{c}{f(c)} & f(c) \\ f'(c) & f'(c) & f'(c) & f'(c) \end{pmatrix}. $$

Then its characteristic equation can be expressed in the form

$$ \lambda^2 - (A + B)\lambda + B = 0 $$

with $A = A(c) := 1 - f(c) > 0$ and $B = B(c) := f'(c)c / f(c) \geq 0$. A direct calculation yields the two eigenvalues of the matrix $J_1$ as

$$ \lambda_+ = \lambda_- := \frac{A + B \pm \sqrt{(A + B)^2 - 4B}}{2} $$

for which, of course, $\lambda_+ + \lambda_- = A + B$ and $\lambda_+ \lambda_- = B$ must hold. Clearly, these two numbers must be proper (i.e. non-real) complex numbers if and only if $(A + B)^2 < 4B$. However, we have

$$ (A + B)^2 < 4B \iff A + B < 2\sqrt{B} \iff B - 2\sqrt{B} + 1 = (1 - \sqrt{B})^2 < 1 - A = f(c) \iff |1 - \sqrt{B}| < \sqrt{f(c)} \iff 1 - \sqrt{B}, \sqrt{B} - 1 < \sqrt{f(c)} \iff 1 - \sqrt{f(c)} < \sqrt{B} < 1 + \sqrt{f(c)} \iff 1 - \sqrt{f(c)} < \frac{f'(c)c}{f(c)} < 1 + \sqrt{f(c)}.$$


Thus, we have \((A + B)^2 < 4B\) if and only if \(1 - \sqrt{\frac{f(c)}{f'(c)}} < \sqrt{\frac{f'(c)c}{f(c)}} < 1 + \sqrt{\frac{f(c)}{f'(c)}}\) or, equivalently, if and only if

\[
1 + f(c) - 2\sqrt{f(c)} < \frac{f'(c)c}{f(c)} < 1 + f(c) + 2\sqrt{f(c)}
\]

holds (which surely always the case if \(B = \frac{f'(c)c}{f(c)} = 1\) holds). So, \(\lambda_+\) and \(\lambda_-\) will form a pair of conjugate complex numbers with modulus \(|\lambda_\pm| = \sqrt{\lambda_+\lambda_-} = \sqrt{B}\) if and only if

\[
1 + f(c) - 2\sqrt{f(c)} < B = \frac{f'(c)c}{f(c)} < 1 + f(c) + 2\sqrt{f(c)}.
\]

Thus, the value of \(B\) determines the local stability of the fixed point \(E\) [Arnold, 1983]: \(E\) is an unstable is an unstable focus in case \(1 < B\) or, equivalently, \(f'(c)c < f(c)\) holds, and it is a stable focus whenever \(1 > B\) or, equivalently, \(f'(c)c > f(c)\) holds.

Remarkably, the same holds if both eigenvalues \(\lambda_\pm\) are real numbers, i.e., in case \((A + B)^2 \geq 4B\) (and, therefore, \(B \neq 1\) in view of \((A + 1)^2 < 4\)). Clearly, assuming now that \(\lambda_\pm \in \mathbb{R}\) holds, both eigenvalues must be non-negative in this case, and at least the larger one of them, i.e., \(\lambda_+\), must be positive as their sum \(A + B\) is positive and their product \(B\) at least non-negative. Also, \(E\) is a stable node in this case if and only if \(B < 1\) holds: Indeed, we have \(0 = \lambda_- < \lambda_+ = A < 1\) in case \(B = 0\), and we have

\[
(A + B)^2 - 4B < (A + B)^2 - 4AB = (A - B)^2
\]

and, therefore, \(\sqrt{(A + B)^2 - 4B} < |A - B|\) in case \(B \neq 0\). Thus,

\[
\min(A, B) = \frac{A + B - |A - B|}{2} < \lambda_- \leq \lambda_+ < \frac{A + B + |A - B|}{2} = \max(A, B)
\]

must hold in this case implying that \(0 < \lambda_- \leq \lambda_+ < 1\) as well \(B = \lambda_- \lambda_+ < \lambda_- < \max(A, B)\) and, therefore, also

\[
B = \min(A, B) = \frac{A + B - |A - B|}{2} < \lambda_- \leq \lambda_+ < \frac{A + B + |A - B|}{2} = \max(A, B) = A
\]

must hold in case \(B < 1\) implying that \(E\) is a stable focus in that case while conversely, if \(1 < B = \lambda_- \lambda_+\) holds, we must have \(\lambda_+ > 1\) as well as

\[
A = \min(A, B) = \frac{A + B - |A - B|}{2} < \lambda_- \leq \lambda_+ < \frac{A + B + |A - B|}{2} = \max(A, B) = B
\]

and, therefore, also \(\lambda_- = B/\lambda_+ > 1\), implying that both eigenvalues of the Jacobian matrix of the map \(\varphi_c\) at \(E\) are larger than 1 in case \(B > 1\).

In conclusion, we get the following theorem regarding the local stability of the fixed point \(E\).

**Theorem 2.** If \(f'(c)c < f(c)\) holds (i.e., if the intercept of the tangent at the point \((c, f(c))\) of the curve (defined by) \(f\) with the y-axis is positive), the fixed point \(E\) is locally asymptotically stable, that is, any orbit \(O^+\) starting from the point in some vicinity of \(E\) eventually approaches this fixed point as \(n\) tends to infinity. Conversely, if \(f'(c)c > f(c)\) holds, the eigenvalues of the Jacobian matrix of the map \(\varphi_c\) at \(E\) either form a pair of complex conjugate numbers of modulus \(> 1\) or a pair of real numbers larger than \(1\), and the fixed point \(E\) is locally unstable, i.e., there is a neighborhood of \(E\) such that the orbit \(O^+ (x(0))\) starting from any point \(x(0)\) within this neighborhood distinct from \(E\) leaves this neighborhood after finitely many iteration steps.
3. The Existence of A Neimark-Sacker Bifurcation

We have seen above that the eigenvalues $\lambda_{\pm}$ become a pair of conjugate complex numbers of modulus

$$|\lambda_{\pm}| = \sqrt{\lambda_{+}\lambda_{-}} = \sqrt{B} = \sqrt{\frac{f'(c)c}{f(c)}}$$

if the inequality $(A + B)^2 < 4B$ or, equivalently,

$$1 - \sqrt{f(c)} < \sqrt{\frac{f'(c)c}{f(c)}} < 1 + \sqrt{f(c)}$$

holds. Furthermore, these complex eigenvalues $\lambda_{\pm}$ are located on the unit circle in the complex plane whenever $B = 1$ or, equivalently, $f'(c)c = f(c)$ holds.

Next, assuming that $f$ is two times differentiable, we may form the derivative of $B(c) = \frac{f'(c)c}{f(c)}$ with respect to $c$:

$$B'(c) = \frac{d}{dc} \left[ \frac{f'(c)c}{f(c)} \right] = f''(c) \frac{c}{f(c)} + f'(c) \left[ \frac{1}{f(c)} - \frac{cf'(c)}{f^2(c)} \right] = f''(c) \frac{c}{f(c)} + f'(c) \frac{f(c) - cf'(c)}{f^2(c)}.$$ 

Thus, the derivative $R'(c)|_{c=\hat{c}}$ of the function $R(c) := \sqrt{B(c)} = \sqrt{\frac{f'(c)c}{f(c)}}$ at any value $c = \hat{c}$ for which $f'(\hat{c})\hat{c} = f(\hat{c})$ and, therefore, $R(c) = 1$ as well as $\frac{\hat{c}}{f(c)} = \frac{f'(c)}{f(c)}$ holds, is

$$R'(c)|_{c=\hat{c}} = \frac{1}{2R(\hat{c})} \frac{f''(\hat{c})\hat{c}}{f(\hat{c})} = \frac{f''(\hat{c})}{2f'(\hat{c})}. $$

In particular, we have $B'(\hat{c}) \neq 0$ and $R'(\hat{c}) \neq 0$ unless $\hat{c}$ is an inflexion point of the function $f$. In addition, as $1 - \frac{f'(\hat{c})}{2}$ is the real part of the eigenvalues $\lambda_{\pm}$, their argument at $\hat{c}$ must be

$$\theta(\hat{c}) := \arccos \left[ 1 - \frac{f'(\hat{c})}{2} \right] \in \left( 0, \frac{\pi}{3} \right).$$

In consequence, we have $e^{i k \theta(\hat{c})} \neq 1$ for $k = 1, 2, 3, 4$ (where $i = \sqrt{-1}$ stands for the imaginary unit).

Thus, together with the discussion of the local (in)stability of the fixed point $E = E(c)$ in the last section and the above argument, the following theorem follows from classical bifurcation theory [Shilnikov et al., 2001; Kuznetsov, 2004; Sacker, 2009].

**Theorem 3.** Assume that the map $f$ is two times differentiable, that $f'(c)c = f(c)$ holds for $c = \hat{c}$ and that $\hat{c}$ is not an inflexion point of $f$, i.e. that $f''(\hat{c}) \neq 0$ holds.

Then, as $c$ passes through $\hat{c}$, the fixed point $E(c)$ of our model (2) changes its stability (either from stable dynamics to unstable dynamics, or from unstable dynamics to stable dynamics) and a unique closed invariant curve $\zeta$ bifurcates from the fixed point, i.e. the orbit $O^+(x(0))$ never leaves off the invariant curve $\zeta$ if it starts from any point $x(0)$ on this curve $\zeta$.

Next, we intend to investigate the stability of the invariant curve $\zeta$. To this end, let us first write the map $\varphi_{\epsilon+\epsilon}$ around $\epsilon = 0$ in the form

$$X \mapsto \Phi_{\epsilon+\epsilon}(X),$$

where the new variable $X$ is defined by $X := x - E(\hat{c} + \epsilon) \in \mathbb{R}^2$. Thus, $X = 0$ becomes the fixed point of the map $\Phi_{\epsilon+\epsilon}$ and, up to terms of higher order, we have

$$\Phi_{\epsilon+\epsilon}(X) \approx |\lambda_{\pm}(\hat{c} + \epsilon)|e^{i \theta(\hat{c} + \epsilon)}X.$$ 

Thus, according to bifurcation theory and the normal-form principle [Shilnikov et al., 2001; Kuznetsov, 2004], the map $\Phi_{\epsilon+\epsilon}$ near the fixed point $X = 0$, with the same assumption as in Theorem 3, must be locally conjugate to the complex normal form:

$$w \mapsto (1 + \beta)e^{i\eta(\beta)}w + \ell(\beta)\overline{w}w^2, \quad w \in \mathbb{C}. \quad (5)$$
Here, $\beta$, as a function of $\hat{c} + \epsilon$, can be regarded as a new real-valued parameter defined by the requirement that $|\lambda_{\pm}(\hat{c} + \epsilon)| = 1 + \beta(\hat{c} + \epsilon)$ holds, and $\eta(\beta)$ is defined by $\eta(\beta) = \theta(\hat{c} + \epsilon)$. Moreover, $\ell(\beta)$ is a complex coefficient in front of the term $\bar{w}w^2$. More specifically, the restriction of the map $\Phi_{\hat{c} + \epsilon}$ to the complex plane at $\epsilon = 0$ is locally conjugate to the normal form:

$$w \mapsto G(w) : w \mapsto e^{i\eta(0)}w + \ell(0)\bar{w}w^2, \quad w \in \mathbb{C},$$

(6)

where $\eta(0) = \theta(\hat{c})$ is the argument of the eigenvalues obtained above so that $\lambda_{\pm}(\hat{c}) = e^{\pm i\theta(\hat{c})}$ holds while $\beta$ vanishes. A more direct illustration of the effect of the normal form can be obtained by utilizing polar coordinates $w = re^{ia}$ in (5). Then, we have

$$re^{ia} \mapsto (1 + \beta)e^{i(\beta)}re^{ia} + \ell(\beta)r^3e^{ia} = (1 + \beta)re^{i\eta(\beta) + a} + \ell(\beta)r^3e^{ia}$$

which can be rewritten into iterative form as follows:

$$r_{n+1}e^{ia_{n+1}} = (1 + \beta)r_ne^{i(\eta(\beta) + a_n)} + \ell(\beta)r_n^3e^{ia}$$

This form, together with the formula $r_{n+1}^2 = r_{n+1}e^{i(\eta(\beta)+a_{n+1})} = r_{n+1}^2 + \bar{r}_{n+1}r_{n+1}^{-1}$, yields:

$$r_{n+1} = r_n \left\{ (1 + \beta)^2 + (1 + \beta) \left[ \ell(\beta)e^{-i(\eta(\beta))} + \bar{\ell}(\beta)e^{i(\eta(\beta))} \right] r_n^2 + \ell(\beta)\bar{\ell}(\beta)r_n^2 \right\}^{1/2}
$$

$$= r_n \left\{ (1 + \beta) + \Re \left[ \ell(\beta)e^{-i(\eta(\beta))} \right] r_n^2 \right\} + O(r_n^3),$$

where the second equality is due to the Taylor series expansion of the square root function. Hence, we can obtain the fixed point $r^* = 0$ and the invariant curve $r^c = \left\{ -\frac{\beta}{\Re \left[ \ell(\beta)e^{-i(\eta(\beta))} \right]} \right\}^{1/2}$ of the approximated iteration:

$$r_{n+1} = r_n \left\{ (1 + \beta) + \Re \left[ \ell(\beta)e^{-i(\eta(\beta))} \right] r_n^2 \right\}.$$
Choose the parameter \( c \) in \((0, 2]\) and consider a sigmoidal function of the form:

\[
f_{\kappa,\mu}(x) = \frac{1}{1 + e^{-\kappa(x - \mu)}},
\]

with \( \kappa := 5 \) and \( \mu \) first assumed to be varying in the interval \([-0.1, 0.8]\). As clearly shown in Fig. 1, the variation of the boundary of the sufficiently evolved orbit \( O^+ \) generated by model (2) with respect to the value of \( f_{\kappa,\mu}(0) \) approximately shows a curve of the form \( O(1) \). Indeed, this is consistent with the estimation of the orbit boundary given in Theorem 1.

Secondly, fix the parameters as \( \kappa := 5 \) and \( \mu := 0.5 \). For simplicity, denote the function \( f_{\kappa,\mu}(x) \) by \( f(x) \) hereafter. Then, the curve \( y = f(x) \) is shown in Fig. 2. In addition, a straight-forward geometric interpretation of the equation \( f'(c)c = f(c) \) yields that the roots of this equation are exactly those values \( x := c \) at which the tangent of the curve \( y = f(x) \) passes through the origin, as shown in Fig. 2. Numerically, these values are \( c_1 \approx 0.2603 \) and \( c_2 \approx 0.6715 \) and, clearly, neither of them is an inflexion point of the function \( f(x) \).

Moreover, Fig. 3 shows the variations of the three quantities \( \sqrt{\frac{f(c)}{f(c)}} + \sqrt{\frac{f(c)}{f(c)}} \), \( \sqrt{\frac{f'(c)c}{f(c)}} \), and \( \sqrt{\frac{f'(c)c}{f(c)}} - \sqrt{f(c)} \) with the parameter \( c \) varying in the interval \((0, 2]\), respectively. This, together with the arguments presented in Section II leading to Theorem 2, implies that the fixed point \( E \) is a stable node with two real eigenvalues that are smaller than 1 provided \( 0 < c \leq c_0 \approx 0.0977 \) holds. However, the fixed point becomes a stable focus with a pair of complex eigenvalues whose moduli are smaller than 1 when either \( c_0 < c \leq c_1 \)
FIG. 3. The variations of three quantities, as indicated in the figure, with the parameter $c$ varying in $(0, 2]$.

FIG. 4. The bottom of the figure is the bifurcation diagram of model (2) for the component $x_1(n)$, when the parameter $c$ is changing in $(0, 2)$ with a step size 0.01. The top three are the phase portraits of model (2) in the $x_1$-$x_2$ plane, corresponding to $c = 0.01$, $c = 0.6$, and $c = 1.6$, respectively. All the initial points are taken as: $x_1(0) = 0.2$ and $x_2(0) = 2.5$.

5. Concluding Remarks

In this paper, we have discussed a simple dynamical system storing, releasing, and replenishing simultaneously some substance in some interdependent way. With the aid of standard techniques, we have shown the existence of a Neimark-Sacker bifurcation with respect to the control parameter of this system. Furthermore, we have provided concrete examples as well as their numerical simulations to illustrate the analytical results we obtained.

Future work along this research direction possibly includes the investigation of

- the synchronizing characteristics as well as the mechanisms of pattern formation in coupled systems arising from coupling a whole array of such systems relative to different coupling configurations [Rulkov, 2001; Hordijk et al., 2010];
• the influence of various types of noise (or perturbations) to the above system and even to the coupled systems [Chen et al., 2007; Lin & Chen, 2006a];
• the much richer dynamical behavior generated by systems where the requirement that the function \( f \) is monotonously increasing is dropped — such as the unimodal function and the sinusoidal function [May, 1976; Lin & Chen, 2006b, 2009].

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Appendix A Computation of \( l(0) \)

Here, we calculate the value of \( l(0) \) to determine the stability of the invariant curve. To this end, restrict the map \( \Phi_{\epsilon}(X) \) at \( \epsilon = 0 \) to the complex plane parameterized by \( w \in \mathbb{C} \):

\[
X = H(w), \quad H : \mathbb{C} \rightarrow \mathbb{R}^2.
\]  

(A.1)

We assume that the function \( H \) can be expanded into

\[
H(w) = qw + \bar{q} \bar{w} + \sum_{j+k=2,3} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k + O(|w|^4),
\]

in which expansion the complex coefficients satisfy \( h_{jk} = \bar{h}_{kj} \), and the vector

\[
q = \left( \frac{-f(\hat{c}) + i \sqrt{4f(\hat{c}) - f^2(\hat{c})}}{2f(\hat{c})f'(\hat{c})}, 1 \right)^{\top}
\]  

(A.2)

is the eigenvector corresponding to the eigenvalue \( e^{i\eta(0)} \), i.e., \( e^{i\eta(0)} q = J_1 q \) holds, and \( \bar{q} \) is the complex conjugate vector of \( q \).

Then, the restricted equation could be written in normal form as in (6) yielding

\[
w \mapsto G(w), \quad G : \mathbb{C} \mapsto \mathbb{C}.
\]  

(A.3)

The substitution of Eqs. (A.1) and (A.3) into Eq. (4) at \( \epsilon = 0 \) gives the equation:

\[
H \circ G(w) = \Phi_{\epsilon} \circ H(w).
\]  

(A.4)

As a matter of fact, we can obtain the accurate form of \( l(0) \) through a comparison of the coefficients in front of the terms of same order on both sides of Equation (A.4). Hence, expanding the function \( \Phi_{\epsilon} \) into multi-variable Taylor series now yields

\[
\Phi_{\epsilon}(X) = J_1 X + \frac{1}{2} J_2(X, X) + \frac{1}{6} J_3(X, X, X) + O(||X||^4),
\]

where \( J_1 \) is the Jacobian matrix defined in (3) and \( J_2 \) and \( J_3 \) are given by

\[
J_2(X, Y) = \begin{pmatrix}
-x_1 y_2 - x_2 y_1, & f''(\hat{c}) f'(\hat{c}) x_1 y_1 + [f'(\hat{c}) + \hat{c} f''(\hat{c})] (x_1 y_2 + x_2 y_1) + \frac{\hat{c}^2 f''''(\hat{c})}{f'(\hat{c})} x_2 y_2
\end{pmatrix}^{\top},
\]

and

\[
J_3(X, Y, Z) = \begin{pmatrix}
0, & f^{(3)}(\hat{c}) f^3(\hat{c}) x_1 y_1 z_1 + [f^{(3)}(\hat{c}) \hat{c} f(\hat{c}) + 2 f''(\hat{c}) f'(\hat{c})] (x_1 y_1 z_2 + x_1 y_2 z_1 + x_2 y_1 z_1) \\
& + \left[ f^{(3)}(\hat{c}) \frac{\hat{c}^2 f''''(\hat{c})}{f'(\hat{c})} + \frac{\hat{c}^3 f''''(\hat{c})}{f'(\hat{c})^3} x_2 y_2 z_2 \right] (x_1 y_2 z_2 + x_2 y_2 z_1 + x_2 y_1 z_2)
\end{pmatrix}^{\top}.
\]  

(A.5)
Now, substituting the above expansions into Equation (A.4) and collecting coefficients of the quadratic terms in this equation, we get

\[ h_{20} = (e^{i\eta(0)} I - J_1)^{-1} J_2(q, q), \]
\[ h_{11} = (I - J_1)^{-1} J_2(q, q), \]  

where \( I \) is a \( 2 \times 2 \) identity matrix. Furthermore, a tedious collection of all the coefficients in front of the \( \overline{w}^2 \)-terms leads to the following equation:

\[ (e^{i\eta(0)} I - J_1) h_{21} = J_3(q, q, q) + 2J_2(q, h_{11}) + J_2(q, h_{20}) - 2\ell(0)q, \]

which is a singular linear system because \( e^{i\eta(0)} \) is the eigenvalue of \( J_1 \). So, it follows from linear algebra that this singular linear system is solvable if and only if

\[ p^T \left[ J_3(q, q, q) + 2J_2(q, h_{11}) + J_2(q, h_{20}) - 2\ell(0)q \right] = 0 \]  

holds where \( p \) is the adjoint eigenvector, i.e., \( e^{-i\eta(0)} p = J_i^T p \) holds. For simplicity, we introduce the normalization condition \( p^T q = 1 \). Then, we get

\[ p = \left( \frac{f'(\hat{c})f(\hat{c})}{2} + i \cdot \frac{f'(\hat{c})(2 - f'(\hat{c}))}{8 - 2f'(\hat{c})} \cdot \sqrt{4f(\hat{c}) - f'(\hat{c})^2}, \frac{1}{2} - i \cdot \frac{\sqrt{4f(\hat{c}) - f'(\hat{c})^2}}{8 - 2f'(\hat{c})} \right)^T, \]

which implies that

\[ \ell(0) = \frac{1}{2} p^T J_3(q, q, q) + p^T J_2(q, h_{11}) + \frac{1}{2} p^T J_2(q, h_{20}) \]  

could be exactly calculated through a substitution of the above-obtained explicit formulas (A.5)-(A.6), and (A.8).

References


REFERENCES


