Nonlocal and multiple-point boundary value problem for fractional differential equations

Wenyong Zhong \textsuperscript{a,b}, Wei Lin \textsuperscript{a,*}

\textsuperscript{a} School of Mathematical Sciences, Research Center and Key Laboratory of Mathematics for Nonlinear Sciences, Fudan University, Shanghai 200433, People's Republic of China
\textsuperscript{b} School of Mathematics and Computer Sciences, Jishou University, Hunan 416000, People's Republic of China

\textbf{A R T I C L E I N F O}

Keywords:
Nonlocal boundary value problem
Caputo's fractional derivative
Fractional integral
Fixed point theorems

\textbf{A B S T R A C T}

In the light of the fixed point theorems, we analytically establish the conditions for the uniqueness of solutions as well as the existence of at least one solution in the nonlocal boundary value problem for a specific kind of nonlinear fractional differential equation. Furthermore, we provide a representative example to illustrate a possible application of the established analytical results.

© 2009 Elsevier Ltd. All rights reserved.

\textbf{1. Introduction}

Fractional calculus is a generalization of ordinary differentiation and integration on an arbitrary order that can be non-integer. This subject, as old as the problem of ordinary differential calculus, can go back to the times when Leibniz and Newton invented differential calculus. As is known to all, the problem for a fractional derivative was originally raised by Leibniz in a letter, dated September 30, 1695. From then on, fractional derivatives have been extensively investigated and then applied theoretically and practically in many fields. In particular, there has been a surge of growth in this subject in the last three decades. Among all the works concerning fractional derivatives, fractional differential equations as an important research branch have attained a great deal of attention from many researchers [1–7].

As one of the focal topics in the research of fractional differential equations, a series of investigations on boundary value problems for some kinds of fractional differential equation with specific configurations have been presented. More specifically, Bai and Lü [8] investigated the existence of positive solutions of the fractional boundary value problem:

\begin{align}
D_{0+}^\alpha u(t) + f(t, u(t)) = 0, & \quad 0 < t < 1, 1 < \alpha \leq 2, \\
\frac{u'(1)}{u(0)} = 0, & \quad u(1) = 0.
\end{align}

(1.1)

where $D_{0+}^\alpha$ represents the standard Riemann–Liouville fractional derivative. With the aid of the Krasnoselskii fixed point theorem and the Leggett–Williams fixed point theorem, they analytically established the criteria on the existence of at least one or three positive solutions for the boundary value problem (1.1).

Later on, Kaufmann and Mboumi [9] discussed the existence of positive solutions for the following fractional boundary value problem:

\begin{align}
D_{0+}^\alpha u(t) + a(t)f(u(t)) = 0, & \quad 0 < t < 1, 1 < \alpha \leq 2, \\
u(0) = 0, & \quad u'(1) = 0.
\end{align}

(1.2)
Their analyses crucially rely on the Krasnoselskii fixed point theorem as well as on the Leggett–Williams fixed point theorem.

Recently, Salem [10] investigated the existence of Pseudo solutions for the nonlinear m-point boundary value problem of fractional type. In particular, he considered the following boundary value problem:

\[
\begin{cases}
D^\alpha x(t) + q(t)f(t, x(t)) = 0, & 0 < t < 1, \ \alpha \in (n - 1, n], \ n \geq 2, \\
x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, & x(1) = \sum_{i=1}^{m-2} \xi_i x(\eta_i),
\end{cases}
\]

(1.3)

where \( x \) takes values in a reflexive Banach space \( E \), \( 0 < \eta_1 < \cdots < \eta_{m-2} < 1 \), and \( \xi_i > 0 \) with \( \sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-1} < 1 \). \( x^{(k)} \) denotes the \( k \)-th Pseudo-derivative of \( x \) and \( D^\alpha \) denotes the Pseudo fractional differential operator of order \( \alpha \). By means of the fixed point theorem attributed to D. O'Regan, a criterion was established for the existence of at least one Pseudo solution for the problem (1.3).

More recently, some mathematicians have considered nonlocal boundary value problems for fractional differential equations [11–13]. In particular, Benchohra, Hamani, and Ntouyas [13] investigated the following nonlocal boundary problem:

\[
\begin{cases}
\mathcal{D}^\alpha u(t) = f(t, u(t)), & 0 < t < T, \ 1 < \alpha \leq 2, \\
u(0) = u(T) = u_T,
\end{cases}
\]

(1.4)

where the operator \( \mathcal{D}^\alpha \) denotes the Caputo’s fractional derivative. Using Schaefer’s fixed point theorem, they provided sufficient criteria for the existence of at least one solution for the problem (1.4) with the conditions that \( f(t, u) \) are uniformly bounded on \([0, T] \times \mathbb{R} \) and that the set \( g(C([0, 1])) \) is bounded. Also, they established criteria for the uniqueness of solutions by virtue of the Banach fixed point theorem.

Motivated by the aforementioned techniques and theorems that are frequently used in coping with boundary value problems for fractional differential equations, we in this paper intend to investigate the possible existence of solutions for the following nonlocal and multiple-point boundary value problem:

\[
\begin{cases}
\mathcal{D}^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, \ 1 < \alpha < 2, \\
u(0) = u_0 + g(u), \quad u'(1) = u_1 + \sum_{i=1}^{m-2} b_i u'(\eta_i).
\end{cases}
\]

(1.5, 1.6)

Denote by \( C([0, 1]) \) the space that contains all continuous functions \( u : [0, 1] \rightarrow \mathbb{R} \), and equip this space with the norm \( \| u \| = \max \{ u(t) : t \in [0, 1] \} \). Thus, \( C([0, 1]) \) with the equipped norm becomes a Banach space. Now, we are in a position to make the following hypotheses, which will be adopted in the following discussion:

(H1) \( f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function.

(H2) There exist a positive constant \( l < 1 \) and a continuous and nondecreasing function \( \phi : [0, \infty) \rightarrow [0, \infty) \) such that \( \phi(z) \leq l z \) and \( |g(u) - g(v)| \leq \phi(\| u - v \|) \) for all \( u, v \in C([0, 1]) \).

(H3) \( u_0, \ u_1 \in \mathbb{R}, \ b_i \geq 0, \ 0 < \xi_i < 1, \ i = 1, 2, \ldots, m-2, \) and \( d = \sum_{i=1}^{m-2} b_i < 1 \).

Note that, with the above settings, Eq. (1.5) with boundary conditions (1.6) not only includes the above-mentioned specific boundary value problems in the literature, but also nontrivially extends the situation to a much wider class of boundary value problems for fractional differential equations.

The remainder of paper is organized as follows. Section 2 preliminarily provides some definitions and lemmas which are crucial to the following discussion. Section 3 gives some sufficient conditions for the uniqueness of solutions and for the existence of at least one solution of Eq. (1.5) with boundary conditions (1.6) by means of the contraction principle in the Banach space and by the fixed point theorem attributed to D. O’Regan, respectively. Finally, a concrete example is provided to illustrate the possible application of the established analytical results.

2. Preliminaries

In this section, we preliminarily provides some definitions and lemmas for fractional derivatives which are useful in the following discussion. These definitions and properties can be found in [2] and [4], and references therein.

**Definition 2.1.** The fractional integral of order \( \alpha > 0 \) of a function \( y : (0, \infty) \rightarrow \mathbb{R} \) is given by

\[
I_{0+}^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) ds,
\]

provided the right side is pointwise defined on \((0, \infty)\).
**Definition 2.2.** The fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \mapsto \mathbb{R}$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t - s)^{n-\alpha-1} y(s) \, ds,$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0, \infty)$.

**Definition 2.3.** For a function $y$ given on the interval $[0, \infty)$, the Caputo’s fractional derivative of order $\alpha > 0$ of $y$ is defined by

$$\overset{c}{D}_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} y^{(n)}(s) \, ds,$$

where $n = [\alpha] + 1$.

The following two lemmas, contained in [14], are fundamental in finding an equivalent integral representation of the boundary value problem (1.5) and (1.6).

**Lemma 2.1.** Let $\alpha > 0$. Then, the fractional differential equation $\overset{c}{D}_{0+}^{\alpha} u(t) = 0$ has a solution $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1}$ with $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n = [\alpha] + 1$.

**Lemma 2.2.** Let $\alpha > 0$. Then, we have

$$I_{0+}^\alpha \overset{c}{D}_{0+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1},$$

with some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ and $n = [\alpha] + 1$.

By Lemmas 2.1 and 2.2, we next present an integral representation of the solution of the boundary value problem for the linearized equation.

**Lemma 2.3.** Let $y \in C([0, 1])$. If hypotheses (H1)–(H3) hold, then the fractional differential equation:

$$\overset{c}{D}_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \quad 1 < \alpha < 2,$$

with boundary conditions

$$u(0) = u_0 + g(u), \quad u'(1) = u_1 + \sum_{i=1}^{m-2} b_i u'(\xi_i),$$

has a unique solution which is given by

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) \, ds + \frac{\alpha - 1}{(1-\alpha)\Gamma(\alpha)} \int_0^1 t(1 - s)^{\alpha-2} y(s) \, ds - \frac{\alpha - 1}{(1-\alpha)\Gamma(\alpha)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} t(\xi_i - s)^{\alpha-2} y(s) \, ds + \frac{1}{1-\alpha} u_1 t + u_0 + g(u).$$

**Proof.** Lemmas 2.1 and 2.2 together yield

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) \, ds + c_1 + c_2 t.$$  

This, with the condition that $u(0) = u_0 + g(u)$, gives $c_1 = u_0 + g(u)$.

Furthermore, differentiation of (2.3) with respect to $t$ produces

$$u'(t) = -\frac{(\alpha - 1)}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-2} y(s) \, ds + c_2,$$

which, with the condition that $u'(1) = u_1 + \sum_{i=1}^{m-2} b_i u'(\xi_i)$, implies

$$-(\alpha - 1) \int_0^1 (1 - s)^{\alpha-2} y(s) \, ds + c_2 = u_1 + \sum_{i=1}^{m-2} b_i \left[ \int_0^{\xi_i} (\xi_i - s)^{\alpha-2} y(s) \, ds + c_2 \right],$$

so that

$$c_2 = \frac{\alpha - 1}{(1-\alpha)\Gamma(\alpha)} \left( \int_0^1 (1 - s)^{\alpha-2} y(s) \, ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-2} y(s) \, ds \right) + \frac{1}{1-\alpha} u_1.$$
Now, substitution of $c_1$ and $c_2$ into (2.3) gives
\[
u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds + \frac{\alpha - 1}{(1-d)\Gamma(\alpha)} \int_0^1 t(1-s)^{\alpha-2} f(s) \, ds
\]
\[-\frac{\alpha}{(1-d)\Gamma(\alpha)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} t(\xi_i - s)^{\alpha-2} f(s) \, ds + \frac{1}{1-d} u_1 t + u_0 + g(u),\]
which verifies the existence of the solution.

The aim of the following argument is to prove the uniqueness of the solution. To this end, assume that $u(t)$ and $v(t)$ are two solutions of the boundary value problem (2.1) and (2.2). Then, analogous to (2.3), we obtain $u(t) - v(t) = c_3 + c_4 t$, where $c_3$ and $c_4$ are pending for estimation. From (2.2) and hypotheses (H2) and (H3), we further have that
\[|c_3| = |u(0) - v(0)| = |g(u) - g(v)| \leq l \max_{t \in [0,1]} |c_3 + tc_4|,\]
and that
\[|c_4| = |u'(1) - v'(1)| \leq \sum_{i=1}^{m-2} b_i |c_4|.
\]
These inequalities, also with hypotheses (H2) and (H3), manifest that $c_4 = 0$ and so $c_3 = 0$. Hence, $u(t) \equiv v(t)$ on $[0,1]$, which completes the proof. \hfill \Box

Next, we introduce the fixed point theorem which was established by O'Regan in [15]. This theorem will be adopted to prove the main results in the following section.

**Lemma 2.4.** Denote by $\mathcal{U}$ an open set in a closed, convex set $C$ of a Banach space $E$. Assume $0 \in \mathcal{U}$. Also assume that $F(\mathcal{U})$ is bounded and that $F : \mathcal{U} \mapsto E$ is given by $F = F_1 + F_2$, in which $F_1 : \mathcal{U} \mapsto E$ is continuous and completely continuous and $F_2 : \mathcal{U} \mapsto E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0, \infty) \mapsto [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \leq \phi(\|x - y\|)$ for all $x, y \in \mathcal{U}$). Then, either
\[(C1) \text{ F has a fixed point } u \in \mathcal{U}; \text{ or}
\[(C2) \text{ there exist a point } u \in \partial \mathcal{U} \text{ and } \lambda \in (0, 1) \text{ with } u = \lambda F(u), \text{ where } \mathcal{U} \text{ and } \partial \mathcal{U}, \text{ respectively, represent the closure and boundary of } \mathcal{U}.
\]

**3. Main results**

In order to utilize the fixed point theorem to solve the boundary value problem specified in (1.5) and (1.6), we first import some notations and operators.

For a given positive number $r$, define the function space $\Omega_r$ by
\[\Omega_r = \{u \in C([0, 1]) : \|u\| < r\},\]
and denote the maximal number by
\[M_r = \max\{|f(t, u)| : (t, u) \in [0, 1] \times [-r, r]\}.\]
Also define three operators from the continuous functions space $C([0, 1])$ to itself, respectively, by
\[
[\mathcal{A}_1 u](t) = \frac{u_1 t}{1-d} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s)) \, ds + \frac{\alpha - 1}{(1-d)\Gamma(\alpha)} \int_0^1 t(1-s)^{\alpha-2} f(s, u(s)) \, ds
\]
\[-\frac{\alpha}{(1-d)\Gamma(\alpha)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} t(\xi_i - s)^{\alpha-2} f(s, u(s)) \, ds,\]
\[
[\mathcal{A}_2 u](t) = u_0 + g(u),\]
\[
[\mathcal{A} u](t) = [\mathcal{A}_1 u](t) + [\mathcal{A}_2 u](t).\]

It is easy to verify that the operator $\mathcal{A}$ is well defined, and that the fixed point of the operator $\mathcal{A}$ is the solution of Eq. (1.5) with boundary conditions (1.6). Furthermore, we have the following lemma on the complete continuity of the operator $\mathcal{A}$.

**Lemma 3.1.** Assume that hypotheses (H1)–(H3) hold. Then, the operator $\mathcal{A} : \hat{\Omega}_r \mapsto C([0, 1])$ is completely continuous.

**Proof.** We first verify that the set $\mathcal{A}_1(\hat{\Omega}_r)$ is bounded. As a matter of fact, from the definition of the operator $\mathcal{A}_1$, we have that, for any $u \in \Omega_r$,
\[ \| A_1 u \| \leq \frac{M_r}{(1 - d) \Gamma(\alpha)} \left( (1 - d) \int_0^1 (1 - s)^{\alpha - 1} \, ds + (\alpha - 1) \int_0^1 (1 - s)^{\alpha - 2} \, ds + (\alpha - 1) \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (s - s)^{\alpha - 2} \, ds \right) + \frac{1}{(1 - d)} |u_1| \]

This clearly validates the uniform boundedness of the set \( A_1(\tilde{\Omega}_r) \).

In addition, for any \( t_1, t_2 \in [0, 1] \), \( t_1 < t_2 \), we have the following estimation:

\[ \| [A_1 u](t_2) - [A_1 u](t_1) \| \leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left( (t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1} \right) |f(s, u(s))| \, ds + \frac{M_r + M_r \sum_{i=1}^{m-2} b_i \xi_i^{\alpha - 1} + |u_1| \Gamma(\alpha)}{(1 - d) \Gamma(\alpha)} |t_2 - t_1| \]

This estimation, with the uniform continuity of the function \( t^{\alpha - 1} \) on \([0, 1]\), leads to a conclusion that the equi-continuity of the elements in the set \( A_1(\tilde{\Omega}_r) \). Therefore, in light of the well-known Arzelà-Ascoli Theorem, we approach that \( A_1(\tilde{\Omega}_r) \) is a relatively compact set. Now, let \( u_n \subset \tilde{\Omega}_r \) with \( |u_n - v| \to 0 \). Then the limit \( \lim_{n \to \infty} u_n = \lim_{n \to \infty} v \) is uniformly valid on \([0, 1]\). From the uniform continuity of \( f(t, u) \) on the compact set \([0, 1] \times [r, \tilde{r}] \), it follows that \( \lim_{n \to \infty} f(t, v_n(t)) = f(t, v(t)) \) is uniformly valid on \([0, 1]\). Hence, it is not hard to verify that \( \| A_1 u_n - A_1 v \| \to 0 \) as \( n \) tends toward positive infinity. As a consequence, we complete the whole proof. \( \square \)

We now present two main results on the uniqueness and existence of the solutions of Eq. (1.5) with boundary conditions (1.6).

**Theorem 3.1.** Assume that hypotheses (H1)–(H3) hold. Also suppose that (H4) there exists a constant \( \Gamma^* > 0 \) satisfying

\[ |f(t, u) - f(t, v)| \leq \Gamma^* |u - v| \quad \text{for each} \quad t \in [0, 1] \quad \text{and all} \quad u, \ v \in \mathbb{R}, \]

and that (H5) the constant \( l \) in assumption (H2) satisfies

\[ \left( 1 - d + \alpha + \alpha \sum_{i=1}^{m-2} b_i \xi_i^{\alpha - 1} \right) \Gamma^* + l < 1. \]

Then, the boundary value problem (1.5) with (1.6) has a unique solution on \([0, 1]\).

**Proof.** From the definition of the operator \( A \) and (H4), we have the following estimations:

\[ \| [Au](t) - [Av](t) \| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} |f(s, u(s)) - f(s, v(s))| \, ds + \frac{\alpha - 1}{(1 - d) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha - 2} |f(s, u(s)) - f(s, v(s))| \, ds \]

\[ + \frac{\alpha - 1}{(1 - d) \Gamma(\alpha)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (s - s)^{\alpha - 2} |f(s, u(s)) - f(s, v(s))| \, ds + |g(u) - g(v)| \]

\[ \leq \left[ \left( 1 - d + \alpha + \alpha \sum_{i=1}^{m-2} b_i \xi_i^{\alpha - 1} \right) \Gamma^* + l \right] \| u - v \|. \]
for any pair of \( u, v \in C([0, 1]) \). Since (H5) is supposed to be valid, the above estimation thus implies that the operator \( A \) is a contraction map from the Banach space \( C([0, 1]) \) to itself. Consequently, the operator \( A \) has a unique fixed point, so that Eq. (1.5) with conditions (1.6) admits a unique solution. \( \square \)

In order to present the next result, we further impose the following hypotheses.

(H6) \( g(0) = 0 \).

(H7) There exists a nonnegative function \( p \in C([0, 1]) \) with \( p > 0 \) on a subinterval of \([0, 1]\). Also there exists a nondecreasing function \( \psi : [0, \infty) \to [0, \infty) \) such that \( |f(t, u)| \leq p(t) \psi(|u|) \) for any \((t, u) \in [0, 1] \times \mathbb{R}\).

(H8) \( \sup_{t \in (0, \infty)} \frac{p_0}{k_0 + p_0 \psi(r_0)} > \frac{1}{\alpha - 1} \), in which

\[
p_0 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} p(s) \, ds + \frac{\alpha - 1}{(1 - d) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-2} p(s) \, ds
\]

\[
+ \frac{\alpha - 1}{(1 - d) \Gamma(\alpha)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-2} p(s) \, ds,
\]

and

\[
k_0 = |u_0| + \frac{|u_1|}{1 - d}.
\]

**Theorem 3.2.** If (H1)–(H3) and (H6)–(H8) hold, then the nonlocal and multiple-point boundary value problem (1.5) with (1.6) has at least one solution.

**Proof.** Take the operator \( A : C([0, 1]) \to C([0, 1]) \) as that defined in (3.3), that is

\[
[A\lambda](t) = [A_1 \lambda](t) + [A_2 \lambda](t),
\]

where the operators \( A_1 \) and \( A_2 \) are the same as those defined in (3.1) and (3.2), respectively.

From (H8), it follows that there exists a number \( r_0 > 0 \) such that

\[
\frac{r_0}{k_0 + p_0 \psi(r_0)} > \frac{1}{\alpha - 1}.
\]

In what follows, we aim to verify the validity of all the conditions in Lemma 2.4 with respect to the operators \( A_1 \), \( A_2 \) and \( A \).

On one hand, it follows from Lemma 3.1 that the operator \( A_1 : \tilde{\Omega}_{r_0} \to C([0, 1]) \) is completely continuous and \( A_1(\tilde{\Omega}_{r_0}) \) is bounded.

On the other hand, hypothesis (H2) implies that the operator \( A_2 : \tilde{\Omega}_{r_0} \to C([0, 1]), A_2(u) = u_0 + g(u) \), is contractive. Moreover, both (H2) and (H6) imply that

\[
\|A_2(u)\| \leq |u_0| + lr_0,
\]

for any \( u \in \tilde{\Omega}_{r_0} \). This, with the boundedness of the set \( A_1(\tilde{\Omega}_{r_0}) \), thus implies that the set \( A(\tilde{\Omega}_{r_0}) \) is bounded.

Finally, it is to show that the case (C2) in Lemma 3.1 does not occur. To this end, we perform the argument by contradiction. Suppose that (C2) holds. Then, we have that there exist \( \lambda \in (0, 1) \) and \( u \in \partial \Omega_{r_0} \) such that \( u = \lambda A\lambda u \). So, we have \( \|u\| = r_0 \) and

\[
u(t) = \lambda \left[ u_0 + g(u) + \frac{u_1 t}{1 - d} - \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, u(s)) \, ds + \frac{\alpha - 1}{(1 - d) \Gamma(\alpha)} \int_0^t (t - s)^{\alpha-2} f(s, u(s)) \, ds
\]

\[
- \frac{\alpha - 1}{(1 - d) \Gamma(\alpha)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-2} f(s, u(s)) \, ds \right].
\]

With hypotheses (H6)–(H8), we further have that

\[
r_0 \leq |u_0| + lr_0 + \frac{|u_1|}{1 - d} + \frac{\psi(r_0)}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} p(s) \, ds
\]

\[
+ \frac{(\alpha - 1) \psi(r_0)}{(1 - d) \Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-2} p(s) \, ds + \frac{(\alpha - 1) \psi(r_0)}{(1 - d) \Gamma(\alpha)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-2} p(s) \, ds,
\]

which implies

\[
r_0 \leq lr_0 + k_0 + p_0 \psi(r_0).
\]
However,  
\[ r_0 \geq \frac{k_0 + p_0 \psi (r_0)}{1 - 1 - l^2} \]
actually contradicts the condition (3.6). Consequently, we have proved that the operators \( A_1, A_2 \) and \( A \) satisfy all the conditions in Lemma 2.4. Thus, we approach a conclusion that \( A \) has at least one fixed point \( u \in \bar{D}_{r_0} \), which is the solution of Eq. (1.5) with nonlocal boundary and multiple-point conditions (1.6). □

4. An illustrative example

In this section, we illustrate the possible application of the above-established analytical results with a concrete example. Let \( \beta > 0 \) and consider the following fractional differential equation:

\[ C D^3 \! \! \! _{5} u(t) = \beta t u(t) \sin u(t), \quad t \in [0, 1], \]  
(4.1)

with the nonlocal and three-point boundary value conditions:

\[ u(0) = \frac{1}{2} + l u(\eta), \quad u'(1) = \frac{1}{2}, \quad 0 < \eta < 1. \]  
(4.2)

Now, we claim that Eq. (4.1) with conditions (4.2) admits at least one solution provided that \( l \) < 1 and \( 0 < \beta < \frac{\sqrt{2}}{\sqrt{2}} (1 - |l|)^2 \). In order to show the validity of this claim, we need to verify that all the conditions in Theorem 3.2 are satisfied.

In fact, note that \( g(u) = l u(\eta) \), the functional \( g \) is contractive because \( |g(u) - g(v)| < |l| \cdot |u - v| \) for any \( u, v \in C([0, 1]) \). Moreover, \( g(0) = 0 \). Hence, the condition (H6) is satisfied. Set \( p(t) = \beta t \) and \( \psi (u) = u^2 \). We have

\[ |f(t, u)| = |\beta t u(t) \sin u(t)| \leq \beta |tu|^2, \]

for any pair \((t, u) \in [0, 1] \times \mathbb{R}\). This verifies the validity of condition (H7) assumed in Theorem 3.2. Finally, note that \( u_0 = \frac{1}{2} \) and \( u_1 = \frac{1}{2} \). Then, we have \( k_0 = 1 \), so that \( p_0 = \frac{8 \beta}{3 \sqrt{2}} \). Consequently, we arrive at the estimation:

\[ \sup_{r \in (0, \infty)} \frac{r}{k_0 + p_0 \psi (r)} = \sup_{r \in (0, \infty)} \frac{r}{1 + \frac{8 \beta}{3 \sqrt{2} r^2}} = \frac{1}{2} \sqrt{\frac{3 \sqrt{2}}{8 \beta}} > \frac{1}{1 - |l|} \]

provided with \( l \) < 1 and \( 0 < \beta < \frac{\sqrt{2}}{\sqrt{2}} (1 - |l|)^2 \). This means that (H5) is satisfied as long as both \( l \) < 1 and \( 0 < \beta < \frac{\sqrt{2}}{\sqrt{2}} (1 - |l|)^2 \) hold. Therefore, according to Theorem 3.2, we can conclude that Eq. (4.1) with the nonlocal and three-point boundary value conditions (4.2) admits at least a solution provided with \( l \) < 1 and \( 0 < \beta < \frac{\sqrt{2}}{\sqrt{2}} (1 - |l|)^2 \).

Acknowledgements

The authors are grateful to the anonymous referee and Prof. Yong Zhou for their significant suggestions on the improvement of this paper. This work was supported by the NNSF of China (Grant Nos. 10501008 and 60874121), and by the Rising-Star Program Foundation of Shanghai, China (Grant No. 07QA14002).

References