Coexistence and local stability of multiple equilibria in neural networks with piecewise linear nondecreasing activation functions

Wang Lili, Lu Wenlian, Chen Tianping

Shanghai Key Laboratory for Contemporary Mathematics, School of Mathematical Sciences, Fudan University, Shanghai 200433, PR China
Key Laboratory of Nonlinear Science of Chinese Ministry of Education, School of Mathematical Sciences, Fudan University, Shanghai 200433, PR China

1. Introduction

In past decades, neural networks have been extensively studied due to their applications in image processing, pattern recognition, associative memories and many other fields. Wilson and Cowan (1972) studied the dynamics of spatially localized neural populations, and introduced two functions $E(t)$ and $I(t)$ to characterize the states of excitatory neurons and inhibitory neurons, respectively. And in Wilson and Cowan (1973), authors derived the following differential equations

$$\begin{align*}
    \frac{d}{dt}(E(x,t)) &= -\langle E(x,t) \rangle + [1 - r_E(E(x,t))]|z_i|\mu|\varphi_E(E(x,t))| \\
    \frac{d}{dt}(I(x,t)) &= -\langle I(x,t) \rangle + [1 - r_I(I(x,t))]|z_i|\mu|\varphi_E(E(x,t))|
\end{align*}$$

where $(E(x,t),\ I(x,t))$ represent time coarse-grained excitatory and inhibitory activities, respectively; $|z_i|$ are the expected proportions of excitatory neurons and inhibitory neurons receiving at least threshold excitation per unit time; $\varphi_j(\cdot)$ stands for the probability that cells of class $j$ be connected with cells of class $i$ a distance $a$ away; $\otimes$ denotes spatial convolution, $P(x,t),Q(x,t)$ are the afferent stimuli to excitatory neurons and inhibitory neurons, respectively. The results obtained are closely related to the biological systems and succeeded in providing qualitative descriptions of several neural processes. Grossberg (1973) introduced another class of recurrent on-center off-surround networks, which were shown to be capable of contrast enhancing significant input information; sustaining this information in short term memory, producing multistable equilibrium points that normalize, or adapt, the field’s total activity, suppressing noise; and preventing saturation of population response even to input patterns whose intensities are high (Ellias & Grossberg, 1975). In such an on-center off-surround anatomy, a given population excites itself and possibly nearby populations and inhibits populations that are further away (possibly itself and nearby populations also). And in Cohen and Grossberg (1983), Cohen–Grossberg neural networks were proposed, which can be described by the following differential equations

$$\frac{du_i}{dt} = a_i(u_i) \left[ b_i(u_i) - \sum_{j=1}^{n} c_{ij}h_j(u_j) \right], \quad i = 1, \ldots, n.$$

In particular, let $a_i(\cdot) \equiv 1$, $b_i(x) = -d_i x$, then, the Cohen–Grossberg neural networks reduce to the following Hopfield neural networks

$$\frac{du_i(t)}{dt} = -d_i u_i(t) + \sum_{j=1}^{n} w_{ij}f_j(u_j(t)) + I_j, \quad i = 1, \ldots, n,$$

where $u_i(t)$ represents the state of the $i$-th unit at time $t$; $d_i > 0$ denotes the rate with which the $i$-th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs; $w_{ij}$ corresponds to the connection weight of...
the \( j \)-th unit on the \( i \)-th unit; \( f_j(\cdot) \) is the activation function; and \( l_i \) stands for the external input.

There has been a large number of works on the dynamics of neural networks in the literature. Note that in many existing works, the authors mainly focused on the existence of a unique equilibrium and its stability, see Chen (2001), Chen and Amari (2001) and other papers. However, in practice, it is desired that the network has several equilibria, of which each represents an individual pattern. For example, in the Cellular Neural Networks (CNNs) with saturated activation function \( f_j(x) = \tanh(x) \), a pattern or an associative memory is usually stored as a binary vector in \([-1, 1]^n\), and the process of the pattern recognition or memory attaining is that the system converges to certain stable equilibrium with all components located in \((-\infty, -1)\) or \((1, +\infty)\). Also in some neuromorphic analog circuits, multistable dynamics even play an essential role, as revealed in Douglas, Koch, Mahowald, Martin, and Suarez (1995), Hahnloser, Sarpeshkar, Mahowald, Douglas, and Seung (2000), and Wersing, Beyn, and Ritter (2001). Therefore, the study of coexistence and stability of multiple equilibrium points, in particular, the attraction basin, is of great interest in both theory and applications.

In an earlier paper (Chen & Amari, 2001), the authors pointed out that the 1-neuron neural network model \( \frac{dx_i}{dt} = -u(t) + (1 + \epsilon) g(u(t)) \), where \( \epsilon \) is a small positive number and \( g(u) = \tanh(u) \), has three equilibrium points and two of them are locally stable, one is unstable. Recently, for the \( n \)-neuron neural networks, many results have been reported in the literature, see Ma and Wu (2007); Remy, Ruet, and Thieffry (2008); Shayer and Campbell (2000); Zeng and Wang (2006); Zhang and Tan (2004) and Zhang, Yi, and Yu (2008). In Zeng and Wang (2006), by decomposing phase space \( \mathbb{R}^n \) into 3\(^2\) subsets, the authors investigated the multiperiodicity of delayed cellular neural networks and showed that the \( n \)-neural networks can have 2\(^n\) stable periodic orbits located in 2\(^n\) subsets of \( \mathbb{R}^n \). The multistability of Cohen–Grossberg neural networks with a common class of piecewise activation functions was also discussed in Cao, Feng, and Wang (2008). It was shown in Cao et al. (2008), Cheng, Lin, and Shi (2007), Zeng and Wang (2006), and other papers that under some conditions, the \( n \)-neuron networks can have 2\(^n\) locally exponentially stable equilibrium points located in 2\(^n\) stability regions. But it is still unknown what happens in the remaining 3\(^n\) – 2\(^n\) subsets. In Cheng, Lin, and Shi (2006), the authors indicated that there can be 3\(^n\) equilibrium points for the \( n \)-neuron neural networks. However, they only gave emphasis on 2\(^n\) equilibrium points which are stable in a class of subsets with positive invariance, never mentioned the stability nor the dynamical behaviors of solutions in other 3\(^n\) – 2\(^n\) subsets. To the best of our knowledge, there are few papers addressing the dynamics in the remaining 3\(^n\) – 2\(^n\) subsets of \( \mathbb{R}^n \), nor the attraction basins of all stable equilibrium points.

In this paper, we investigate the neural networks (1) and deal with these issues. To be more general, we present a class of nondecreasing piecewise linear activation functions with 2\(r\) corner points, which can be described by:

\[
\begin{align*}
  f_j(x) = & \begin{cases} 
    m_j^1, & -\infty < x < p_j^1, \\
    \frac{m_j^2 - m_j^1}{q_j^1 - p_j^1}(x - p_j^1) + m_j^1, & p_j^1 \leq x \leq q_j^1, \\
    m_j^2, & q_j^1 < x < q_j^2, \\
    \frac{m_j^3 - m_j^2}{q_j^2 - p_j^2}(x - p_j^2) + m_j^2, & p_j^2 \leq x \leq q_j^2, \\
    m_j^3, & q_j^2 < x < p_j^3, \\
    \ldots & \ldots \ldots \ldots \\
    m_j^{r+1} - m_j^r, & q_j^r < x < +\infty, \\
    \frac{m_j^{r+1} - m_j^r}{q_j^r - p_j^r}(x - p_j^r) + m_j^r, & p_j^r \leq x \leq q_j^r, \\
    m_j^{r+1}, & q_j^r < x < +\infty
  \end{cases}
\end{align*}
\]

where \( r \geq 1 \). \( [m_j^k]_{k=1}^{r+1} \) is an increasing constant series, \( p_j^k, q_j^k \), \( k = 1, 2, \ldots, r \) are constants with \(-\infty < p_j^1 < q_j^1 < p_j^2 < q_j^2 < \ldots \ldots < p_j^r < q_j^r < +\infty, j = 1, 2, \ldots, n.\)

The neural networks with activation function (2) can store many more patterns or associative memories than those with saturated function, it is meaningful in applications Fig. 2.

In the following, we will precisely figure out all equilibria for the system (1), and investigate the stability and attraction basin for each equilibrium. Discussions and Simulations are also provided to illustrate and verify theoretical results.

We begin with the multistability for \( r = 1 \).

2. Case I: \( r = 1 \)

In this case, the activation function \( f_j(\cdot) \) reduces to

\[
\begin{align*}
  f_j(x) = & \begin{cases} 
    m_j, & -\infty < x < p_j, \\
    \frac{M_j - m_j}{q_j - p_j}(x - p_j) + m_j, & p_j \leq x \leq q_j, \\
    M_j, & q_j < x < +\infty
  \end{cases}
\end{align*}
\]
where $m_j, M_j, p_j, q_j$ are constants with $m_j < M_j, p_j < q_j, j = 1, 2, \ldots, n$. And we first investigate 2-neuron neural networks. The $n$-neuron neural networks can be dealt with similarly.

2.1. Neural networks with 2-neurons

In this case, the network (1) reduces to

$$\begin{align*}
\frac{du_1(t)}{dt} &= -d_1 u_1(t) + w_{11} f_1(u_1(t)) + w_{12} f_2(u_2(t)) + I_1, \\
\frac{du_2(t)}{dt} &= -d_2 u_2(t) + w_{21} f_1(u_1(t)) + w_{22} f_2(u_2(t)) + I_2.
\end{align*}$$

and we prove

**Theorem 1.** Suppose that

$$\begin{align*}
-d_1 p_1 + w_{11} m_1 + \max\{w_{11} m_j, w_{12} M_j\} + I_1 &< 0, \\
-d_2 q_2 + w_{21} M_1 + \min\{w_{21} m_j, w_{22} M_j\} + I_2 &> 0,
\end{align*}$$

for $i, j = 1, 2, i \neq j$. Then system (4) has 9 equilibrium points, in which all 4 are locally exponentially stable and others are unstable.

**Proof.** Obviously, $R^2$ can be divided into 9 subsets (see Fig. 3):

$S_1 = (-\infty, p_1) \times (q_2, \infty), \\
S_2 = (q_1, \infty) \times (q_2, \infty), \\
S_3 = (-\infty, p_1) \times (p_2, q_2), \\
S_4 = (q_1, \infty) \times (p_2, q_2), \\
S_5 = (-\infty, p_1) \times (-\infty, p_2), \\
S_6 = (q_1, q_2) \times (-\infty, p_2), \\
S_7 = (q_1, q_2) \times (q_2, \infty), \\
S_8 = (-\infty, p_1) \times (-\infty, q_2), \\
S_9 = (-\infty, p_1) \times (-\infty, q_2).

We discuss each subset separately.

(i) Subset $S_1 = (-\infty, p_1) \times (q_2, \infty)$

In this subset, any equilibrium is a root of the following equations

$$\begin{align*}
-d_1 u_1 + w_{11} m_1 + w_{12} M_2 + I_1 &= 0, \\
-d_2 u_2 + w_{21} m_1 + w_{22} M_2 + I_2 &= 0.
\end{align*}$$

By (5), we know that

$$\begin{align*}
-d_1 p_1 + w_{11} m_1 + w_{12} M_2 + I_1 &< 0, \\
\lim_{x \to +\infty} (-d_1 x + w_{11} m_1 + w_{12} M_2 + I_1) &= +\infty, \\
-d_2 q_2 + w_{21} m_1 + w_{22} M_2 + I_2 &> 0, \\
\lim_{x \to +\infty} (-d_2 x + w_{21} m_1 + w_{22} M_2 + I_2) &= -\infty.
\end{align*}$$

Therefore, (6) has a unique solution $u^* = (u_1^*, u_2^*) \in S_1$, which is also the unique equilibrium point of (4) in $S_1$.

Let $u(t)$ be a solution of dynamical system (4) with initial state $(u_1(0), u_2(0)) \in S_1$. We claim that $u(t)$ will stay in $S_1$ for all $t > 0$. In fact, from conditions (5), we can find a small constant $\epsilon > 0$ such that

$$\begin{align*}
-d_1 (p_1 - \epsilon) + w_{11} m_1 + w_{12} M_2 + I_1 &< 0, \\
-d_2 (q_2 + \epsilon) + w_{21} m_1 + w_{22} M_2 + I_2 &> 0.
\end{align*}$$

If for some $t_1 > 0, (u_1(t_1), u_2(t_1)) \in S_1$ such that $u_1(t_1) > p_1 - \epsilon$ (or $u_2(t_1) < q_2 + \epsilon$), then, we have $\frac{du_1(t)}{dt} |_{t=t_1} < 0$ (or $\frac{du_2(t)}{dt} |_{t=t_1} > 0$), which implies the solution $u(t) \in S_1$ for all $t > 0$.

Define $x_i(t) = u_i(t) - u_i^*$, $i = 1, 2$. Then, we have

$$\begin{align*}
\frac{dx_1(t)}{dt} &= -d_1 x_1(t), \\
\frac{dx_2(t)}{dt} &= -d_2 x_2(t).
\end{align*}$$

Therefore, $u^*$ is locally stable.

Similarly, we conclude that in each subset $S_2, S_3, S_4$, there is a unique equilibrium point, which is locally exponentially stable.

(ii) Subset $\Lambda = [p_1, q_1] \times [p_2, q_2]$

Every equilibrium in subset $\Lambda$ is a root of the following equations

$$\begin{align*}
-d_1 u_1 + w_{11} u_1 u_2 + w_{12} u_2 + \hat{I}_1 &= 0, \\
-d_2 u_2 + w_{21} u_1 u_2 + w_{22} u_2 + \hat{I}_2 &= 0.
\end{align*}$$

where

$$\begin{align*}
l_j &= \frac{M_j - m_j}{q_j - p_j}, \\
c_j &= \frac{m_j q_j - M_j p_j}{q_j - p_j}, \\
i, j &= 1, 2,
\end{align*}$$

and

$$\begin{align*}
\hat{I}_1 &= w_{11} c_1 + w_{12} c_2 + \hat{I}_1, \\
\hat{I}_2 &= w_{21} c_1 + w_{22} c_2 + \hat{I}_2.
\end{align*}$$

In this case, conditions (5) become

$$\begin{align*}
(-d_1 + w_{11}/l_1) p_1 + l_1 \max\{w_{11} p_j, w_{12} q_j\} + \hat{I}_1 &< 0, \\
(-d_2 + w_{11}/l_1) q_1 + l_1 \min\{w_{11} p_j, w_{12} q_j\} + \hat{I}_1 &> 0,
\end{align*}$$

for $i, j = 1, 2, i \neq j$.

If $w_{21} = 0$, by solving the Eq. (10), we get $u_2^* = -\frac{\hat{I}_2}{-d_2 + w_{22} l_2} \in (p_2, q_2)$ and $u_1^* = \frac{w_{21} l_2 (-d_2 + w_{22} l_2)}{(-d_1 + w_{11} l_1) (-d_2 + w_{22} l_2)} \in (p_1, q_1)$. Hence, $(u_1^*, u_2^*)$ is the unique root of Eq. (10) in $\Lambda$.

Otherwise, assume $w_{21} > 0$. Consider the following two straight lines deduced by the Eq. (10):

$$\begin{align*}
y_1(u_2) &= -\frac{w_{12} l_2}{-d_1 + w_{11} l_1} u_2 - \frac{\hat{I}_1}{-d_1 + w_{11} l_1}, \\
y_2(u_2) &= -\frac{d_2 + w_{22} l_2}{w_{21} l_1} u_2 - \frac{\hat{I}_2}{w_{21} l_1}.
\end{align*}$$

By (13), we have

$$\begin{align*}
(-d_1 + w_{11} l_1) q_1 + w_{12} l_2 p_2 + \hat{I}_1 &> 0, \\
(-d_2 + w_{22} l_2) p_2 + w_{21} l_1 q_1 + \hat{I}_2 &< 0.
\end{align*}$$

Eliminating $q_1$, we have

$$\begin{align*}
-\frac{w_{12} l_2}{-d_1 + w_{11} l_1} p_2 - \frac{\hat{I}_1}{-d_1 + w_{11} l_1} < -\frac{d_2 + w_{22} l_2}{w_{21} l_1} p_2 - \frac{\hat{I}_2}{w_{21} l_1},
\end{align*}$$

which means $y_1(p_2) < y_2(p_2)$. 

---

**Fig. 3.** The phase plane diagram of subsets of $R^2$ with $r = 1$. 

---

$L. Wang et al. / Neural Networks 23 (2010) 189–200$
By similar arguments, we have $y_1(q_2) > y_2(q_2)$, too.

Therefore, there must be a unique $u_i^{**} \in (p_2, q_i)$ such that $y_1(u_i^{**}) = y_2(u_i^{**})$. Denote $u_i^{**} = y_1(u_i^{**})$. Then $(u_i^{**}, u_j^{**})$ is the unique solution of (10), which is the unique equilibrium of (4) in $A$.

Let $u(t)$ be a solution of dynamical system (4) with initial state $(u_1(0), u_2(0))$ near $u^{**}$, and $x(t) = u_i(t) - u_i^{**}$, $i = 1, 2$. Then, we have

$$\frac{dx_1(t)}{dt} = (-d_1 + w_{11}l_1)x_1(t) + w_{12}l_2x_2(t),$$
$$\frac{dx_2(t)}{dt} = (-d_2 + w_{22}l_2)x_2(t) + w_{21}l_1x_1(t).$$

(17)

Denote

$$A = \begin{pmatrix} -d_1 + w_{11}l_1 & w_{12}l_2 \\ w_{21}l_1 & -d_2 + w_{22}l_2 \end{pmatrix}. $$

(18)

Because the trace

$$\text{trace}(A) = (-d_1 + w_{11}l_1) + (-d_2 + w_{22}l_2) > 0$$

which implies $\text{Re}(\lambda) > 0$ for at least one eigenvalue $\lambda$ of $A$. Thus, $u^{**}$ is an unstable equilibrium.

(iii) Subset $\hat{S}_1 = [p_1, q_1] \times (q_2, \infty)$

In this case, any equilibrium is a root of the following equations

$$-d_1u_1 + w_{11}u_1 + w_{12}q_2 + \hat{l}_1 = 0,$$
$$-d_2u_2 + w_{21}u_1 + w_{22}q_2 + \hat{l}_2 = 0,$$

(19)

where $\hat{l}_1, \hat{l}_2$ are defined as (12).

Using conditions (5), by similar method, we can prove that $(u_1^{***}, u_2^{***}) \in \hat{S}_1$ is a unique solution (19) and is unstable.

Similar discussions apply to the subsets $\hat{S}_2, \hat{S}_3, \hat{S}_4$.

In summary, there are 3^2 equilibria of system (4) under conditions (5), and 2^3 of them are locally exponentially stable while others are unstable. □

2.2. Neural networks with $n$-neurons

Let

$$(\infty, p_k) = (\infty, p_k)^1 \times [p_k, q_k]^0 \times (q_k, +\infty)^0,$$

$$(p_k, q_k) = (\infty, p_k)^0 \times [p_k, q_k]^1 \times (q_k, +\infty)^0,$$

$$(q_k, +\infty) = (\infty, p_k)^0 \times [p_k, q_k]^0 \times (q_k, +\infty)^1,$$

the phase space $\mathbb{R}^n$ is divided into 3^n subsets (see Fig. 4):

$$\prod_{k=1}^{n} (\infty, p_k)^{s_k} \times [p_k, q_k]^{s_k} \times (q_k, +\infty)^{s_k},$$

$$\delta_k = (1, 0, 0), \text{ or } (0, 1, 0), \text{ or } (0, 0, 1),$$

$k = 1, \ldots, n$.

Furthermore, denote

$$\Phi_1 = \bigcup_{s_k \in \{0, 1\}, k=1, \ldots, n} \left( \prod_{k=1}^{n} (\infty, p_k)^{s_k} \times [p_k, q_k]^{s_k} \times (q_k, +\infty)^{1-s_k} \right),$$

$$\Phi_2 = \prod_{k=1}^{n} [p_k, q_k], \Phi_3 = \mathbb{R}^n - \Phi_1 - \Phi_2.$$

Now, we will prove the following theorem:

**Theorem 2.** Suppose that

$$-d_{pi}u_i + w_{i1}m_i + \sum_{j=1, j \neq i}^{n} \max\{w_{ij}m_j, w_{ij}M_j\} + l_i < 0,$$
$$-d_{qi}u_i + w_{q1}M_i + \sum_{j=1, j \neq i}^{n} \min\{w_{iq}m_j, w_{iq}M_j\} + l_i > 0,$$

(20)

for $i, j = 1, 2, \ldots, n$. Then, the dynamical system (1) has $3^n$ equilibrium points in all, $2^3$ of which are locally exponentially stable and others are unstable.

**Proof.** Similar to the proof of Theorem 1, we discuss the dynamics for every subset, respectively.

(i) Subset $\Phi_1$

Pick

$$\tilde{\Omega}_1 = \prod_{k \in N_1} (\infty, p_k) \times \prod_{k \in N_2} (q_k, +\infty) \subset \Phi_1,$$

where $N_1, N_2$ are subsets of $\{1, 2, \ldots, n\}$, and $N_1 \cup N_2 = \{1, 2, \ldots, n\}, N_1 \cap N_2 = \emptyset$.

Denote

$$F_i(\xi) = -d_{pi}\xi + \sum_{j \in N_1} w_{ij}m_j + \sum_{j \in N_2} w_{ij}M_j + l_i, \ i = 1, \ldots, n.$$

Then, from conditions (20), it is easy to see that

$$F_i(\infty) = +\infty, \ F_i(p_k) < 0, \text{ for } i \in N_1;$$
$$F_i(q_k) > 0, \ F_i(+\infty) = -\infty, \text{ for } i \in N_2.$$

Therefore, the equation set $F_i(\xi) = 0, \ i = 1, \ldots, n$, has a unique root, which is the very equilibrium of system (1) in $\tilde{\Omega}_1$. Denote it as $v^* = (v_1^*, v_2^*, \ldots, v_n^*)$.

Using the same method in the proof of case (i) in Theorem 1, we can prove that $\tilde{\Omega}_1$ is invariant with respect to the solution with initial state in $\tilde{\Omega}_1$. That is, if $u(0) \in \tilde{\Omega}_1$, then the solution $u(t)$ ($t \geq 0$) of the dynamical system (1) will stay in $\tilde{\Omega}_1$. Letting $x(t) = u(t) - v^*$, we have

$$\frac{dx_i(t)}{dt} = -d_{pi}x_i(t), \ i = 1, \ldots, n,$$

(21)

which implies $v^*$ in $\tilde{\Omega}_1$ is stable exponentially.

Because $\Omega_1 \subset \Phi_1$ is chosen arbitrarily. We conclude that in each of the subsets of $\Phi_i$, there is a unique locally exponentially stable equilibrium. Therefore, the system (1) has $2^n$ locally exponentially stable equilibrium points.

(ii) Subset $\Phi_2$

Denote $\Omega_2 = \Phi_2 = \prod_{k=1}^{n} [p_k, q_k]$. Consider the following equations

$$-d_{pi}u_i + w_{i1}u_i + \sum_{j \neq i}^{n} w_{ij}u_j + \sum_{j=1}^{n} w_{ij}c_j + l_i = 0,$$

(22)

where

$$l_j = \frac{M_j - m_j}{q_j - p_j}, \ c_j = \frac{m_jq_j - M_jp_j}{q_j - p_j}, \ i, j = 1, \ldots, n.$$

(23)
For any $u = (u_1, u_2, \ldots, u_n) \in \hat{\Omega}_2$, fix $u_2, \ldots, u_n$, we can find a unique $\tilde{u}_1 \in (p_1, q_1)$ such that
\[-d_1\tilde{u}_1 + w_{11}\tilde{u}_1 + \sum_{j \neq 1} w_{j1}\tilde{u}_j + \sum_{j=1}^n w_{j1}c_j + l_1 = 0,
\]
and since
\[
\begin{align*}
-d_1p_1 + w_{11}p_1 + \sum_{j \neq 1} w_{j1}p_j + \sum_{j=1}^n w_{j1}c_j + l_1 &< 0, \\
d_1q_1 + w_{11}q_1 + \sum_{j \neq 1} w_{j1}q_j + \sum_{j=1}^n w_{j1}c_j + l_1 &> 0.
\end{align*}
\]
Similarly, fixing $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n$ ($i = 2, \ldots, n$), we can find a unique $\tilde{u}_i \in (p_i, q_i)$ such that
\[-d_i\tilde{u}_i + w_{ii}\tilde{u}_i + \sum_{j \neq i} w_{ij}\tilde{u}_j + \sum_{j=1}^n w_{ij}c_j + l_i = 0,
\]
and since
\[
\begin{align*}
-d_ip_i + w_{ii}p_i + \sum_{j \neq i} w_{ij}p_j + \sum_{j=1}^n w_{ij}c_j + l_i &< 0, \\
d_q + w_{ii}q_i + \sum_{j \neq i} w_{ij}q_j + \sum_{j=1}^n w_{ij}c_j + l_i &> 0.
\end{align*}
\]
Define a map
\[T : \hat{\Omega}_2 \to \hat{\Omega}_2 \quad \text{by} \quad (u_1, u_2, \ldots, u_n) \mapsto (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n).
\]
It is clearly a continuous map. By Brouwer’s fixed point theorem, there exists at least one $v^{**} \in \hat{\Omega}_2$ such that $Tv^{**} = v^{**}$, which is also the equilibrium point of (1) in $\hat{\Omega}_2$.

Now, we claim that the coefficient matrix of (22)
\[
\tilde{A} = \begin{pmatrix}
-d_1 + w_{11} & w_{12} & \cdots & w_{1n} \\
-w_{21} & -d_2 + w_{22} & \cdots & w_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-w_{n1} & w_{n2} & \cdots & -d_n + w_{nn}
\end{pmatrix}, \quad (24)
\]
is invertible.

Otherwise, assume that $\tilde{A}$ is not invertible. Denote $\alpha_i$ as its $i$-th row vector. Then, there must exist nonzero constants $k_1, \ldots, k_m$ ($m \leq n - 1$), and $i_1, \ldots, \bar{i}_m$, $m_{n+1} \in \{1, 2, \ldots, n\}$, such that
\[
\alpha_{m_{n+1}} = k_1\alpha_{i_1} + \cdots + k_m\alpha_{\bar{i}_m}, \quad (25)
\]
Without loss of generality, let $k_1, \ldots, k_m > 0$, while $k_{m+1}, \ldots, k_m < 0$. Denote $\Theta$ as the index set such that $\{i_1, \ldots, \bar{i}_m, m_{n+1}\}$.

From conditions (20), it follows that
\[
\begin{align*}
(-d_k + w_{kk}k_k)p_k &+ \sum_{j=1,j \neq k}^s w_{kj}k_jp_j + \sum_{j=m_{n+1}}^{m+1} w_{kj}q_j + \sum_{j=m_{n+1}}^{m+1} w_{kj}c_j + j_k < 0, \\
&\quad k = i_1, \ldots, i_s, \\
(-d_k + w_{kk}k_k)q_k &+ \sum_{j=m_{n+1}}^{m+1} w_{kj}q_j + \sum_{j=1,j \neq k}^s w_{kj}p_j + \sum_{j=m_{n+1}}^{m+1} w_{kj}c_j + j_k > 0, \\
&\quad k = \bar{i}_{m+1}, \ldots, \bar{i}_m,
\end{align*}
\]
where
\[j_k = \sum_{h \in \Theta} w_{hk}p_h + \sum_{h=1}^n w_{hk}c_h + k_h \quad k = i_1, \ldots, \bar{i}_m.
\]
Multiplying each inequality by the corresponding $k_j$ and adding them together, we get
\[
\sum_{j=1}^s \left[-k_dp_j + \sum_{h=1}^m w_{jh}k_h \right]p_j + \sum_{j=m_{n+1}}^{m+1} \left[-k_dp_j + \sum_{h=1}^m w_{jh}k_h \right]q_j,
\]
\[
\sum_{j=m_{n+1}}^{m+1} k_j k_j < 0. \quad (26)
\]
By (25), it is equivalent to
\[
\begin{align*}
(-d_{m_{n+1}} + w_{m_{n+1}m_{n+1}} k_{m_{n+1}})q_{m_{n+1}} + \sum_{j=1}^s w_{m_{n+1}j}q_j + \sum_{j=1}^s w_{m_{n+1}j}p_j + \sum_{j=m_{n+1}}^{m+1} j_k k_j < 0. \quad (27)
\end{align*}
\]
Then, compared with (20), we have
\[
\sum_{j=1}^m j_k k_j < \sum_{h \in \Theta} w_{hm+1}k_hp_h + \sum_{h=1}^n w_{hm+1}c_h + l_{m_{n+1}}. \quad (28)
\]
On the other hand, from conditions (20), it also follows that
\[
\begin{align*}
(-d_k + w_{kk}k_k)p_k &+ \sum_{j=1,j \neq k}^s w_{kj}k_jp_j + \sum_{j=m_{n+1}}^{m+1} w_{kj}q_j + \sum_{j=m_{n+1}}^{m+1} w_{kj}c_j + j_k < 0, \\
&\quad k = i_1, \ldots, i_s, \\
(-d_k + w_{kk}k_k)q_k &+ \sum_{j=m_{n+1}}^{m+1} w_{kj}q_j + \sum_{j=1,j \neq k}^s w_{kj}p_j + \sum_{j=m_{n+1}}^{m+1} w_{kj}c_j + j_k < 0, \\
&\quad k = \bar{i}_{m+1}, \ldots, \bar{i}_m,
\end{align*}
\]
Multiplying each inequality by the corresponding $k_j$ and adding them together, we get
\[
\sum_{j=1}^s \left[-k_dp_j + \sum_{h=1}^m w_{jh}k_h \right]p_j + \sum_{j=m_{n+1}}^{m+1} \left[-k_dp_j + \sum_{h=1}^m w_{jh}k_h \right]q_j,
\]
\[
\sum_{j=m_{n+1}}^{m+1} k_j k_j > 0. \quad (29)
\]
which is equivalent to
\[
\begin{align*}
(-d_{m_{n+1}} + w_{m_{n+1}m_{n+1}} k_{m_{n+1}})p_{m_{n+1}} + \sum_{j=1}^s w_{m_{n+1}j}q_j + \sum_{j=1}^s w_{m_{n+1}j}p_j + \sum_{j=m_{n+1}}^{m+1} j_k k_j > 0. \quad (30)
\end{align*}
\]
Compared with (20), it holds that
\[
\sum_{j=1}^m j_k k_j > \sum_{h \in \Theta} w_{hm+1}k_hp_h + \sum_{h=1}^n w_{hm+1}c_h + l_{m_{n+1}}, \quad (31)
\]
which is a contradiction to (28). Therefore, $\tilde{A}$ is invertible. It means the equilibrium $v^{**}$ of (1) in $\hat{\Omega}_2$ is unique.

Furthermore, let $u(t)$ be a solution of system (1) with initial state nearby $v^{**}$. Denote $x_i(t) = u_i(t) - v_i^{**}, \ i = 1, \ldots, n$. We have
\[
\frac{dx_i(t)}{dt} = (-d_i + w_{ii}l_i)x_i(t) + \sum_{j \neq i} w_{ij}x_j(t), \quad i = 1, \ldots, n. \quad (32)
\]
Because the trace
\[
\text{trace}(\tilde{A}) = \sum_{i=1}^n (-d_i + w_{ii}l_i) > 0.
\]
We conclude that there must be an eigenvalue $\lambda$ of $\tilde{A}$ such that $\text{Re}(\lambda) > 0$. Thus, the system (32) is unstable, so is the equilibrium $\upsilon^{**}$.

(iii) Subset $\Phi_3$

Pick

$$\tilde{\Omega}_3 = \prod_{k \in \mathbb{N}_1} (-\infty, p_k) \times \prod_{k \in \mathbb{N}_2} (q_k, +\infty) \times \prod_{k \in \mathbb{N}_3} [p_k, q_k] \subset \Phi_3,$$

where $\mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3 = \{1, 2, \ldots, n\}$, $\mathbb{N}_1 \cap \mathbb{N}_2 = \emptyset$, $i \neq j$.

Consider the following equations:

$$-d_i u_i + \sum_{j \in \mathbb{N}_1} w_{ij} m_j + \sum_{j \in \mathbb{N}_2} w_{ij} M_j + \sum_{j \in \mathbb{N}_3} w_{ij} c_j + I_i = 0,$$

$i \in \mathbb{N}_1 \cup \mathbb{N}_2$;

$$-d_i u_i + \sum_{j \in \mathbb{N}_1} w_{ij} m_j + \sum_{j \in \mathbb{N}_2} w_{ij} M_j + \sum_{j \in \mathbb{N}_3} w_{ij} c_j + I_i = 0,$$

$i \in \mathbb{N}_1 \cup \mathbb{N}_2$.

Similar to the proof of case (ii), we define a map

$$\tilde{T} : \prod_{j \in \mathbb{N}_1} [p_j, q_j] \rightarrow \prod_{j \in \mathbb{N}_1} [p_j, q_j]$$

$$\left(u_1, u_2, \ldots, u_n\right) \rightarrow \left(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n\right),$$

in which $\mathbb{N}_3$ is denoted as $\mathbb{N}_3 = \{i_1, \ldots, i_k\}$, and $\tilde{u}_i$, $i = 1, \ldots, s$ are defined in the same way as above $\tilde{u}_i$ in case (ii).

Since the map $\tilde{T}$ is continuous, it follows from Brouwer’s fixed point theorem that there exists $(\upsilon^{**}_1, \upsilon^{**}_2, \ldots, \upsilon^{**}_u) \in \prod_{j \in \mathbb{N}_1} [p_j, q_j]$ such that

$$\tilde{T} (\upsilon^{**}_1, \upsilon^{**}_2, \ldots, \upsilon^{**}_u) = (\upsilon^{**}_1, \upsilon^{**}_2, \ldots, \upsilon^{**}_u).$$

By similar arguments used in the proof of case (ii), the coefficient matrix of (34) is invertible, which implies that $(\upsilon^{**}_1, \upsilon^{**}_2, \ldots, \upsilon^{**}_u)$ is unique. Then, replacing $u_j, j \in \mathbb{N}_2$, by $\upsilon^{**}_j$, and solving (33), we obtain $\upsilon^{**}_i$ for $i \in \mathbb{N}_1 \cup \mathbb{N}_2$. Denote $\upsilon^{**} = (\upsilon^{**}_1, \ldots, \upsilon^{**}_u)$. It is just the equilibrium in $\tilde{\Omega}_2$. By the same proof of the case (ii), we conclude that it is unstable.

Because $\tilde{\Omega}_2 \subset \Phi_3$ is chosen arbitrarily, we conclude that in each subset of $\Phi_3$, there is an equilibrium point, which is unstable.

In summary, system (1) has in total 3$^n$ equilibrium points, 2$^n$ of them are locally exponentially stable and others are unstable. $\Box$

3. Case II: $r \geq 1$

Inspired by the discussions above, in this section, we discuss the dynamical system (1) with activation functions (2).

Theorem 3. Suppose that

$$\begin{align*}
-d_i p_i^k + \sum_{j \neq i} m_j &+ \max \{w_{ij} m_j, w_{ij} m_j^{+1}\} + I_i < 0, \\
-d_i q_i^k + \sum_{j \neq i} m_j &+ \min \{w_{ij} m_j, w_{ij} m_j^{+1}\} + I_i > 0,
\end{align*}$$

for $i, j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, r$. Then, the dynamical system (1) has (2$r + 1)^n$ equilibrium points. Among them, $(r + 1)^n$ locally stable and others are unstable.

Proof. Obviously, $\mathbb{R}^n$ can be divided into $(2r + 1)^n$ subsets, so that $\mathbb{R}^n$ can be divided into $(2r + 1)^n$ subsets. For example, when $r = 2$, $\mathbb{R}^n$ can be divided into the following 25 subsets:

In this case, we can prove that there are 25 equilibrium points. Among them, 9 equilibrium points located in subsets $S_i$, $i = 1, \ldots, 9$ are locally stable. Others are unstable. The detailed proof is omitted (Fig. 5). $\Box$

4. Attraction basins of equilibria

In this section, we investigate attraction basins of equilibria for the system (1) with activation functions (2).

We begin with the case $n = 2$ and $r = 1$. Under conditions (5), Theorem 1 tells us that there are 4 locally stable equilibrium points $u^{(1)}$, $u^{(2)}$, $u^{(3)}$, $u^{(4)}$, in subsets $S_1$, $S_2$, $S_3$, $S_4$, respectively, and 5 unstable equilibrium points $u^{(5)}$, $u^{(6)}$, $u^{(7)}$, $u^{(8)}$, $u^{(9)}$ in $S_1$, $S_2$, $S_3$, $S_4$, $\Lambda$, respectively.

Take a look at the dynamics in subsets $S_1$, $S_2$, $S_3$, $S_4$, for example, in $S_1 = [p_1, q_1] \times (q_2, \infty)$, system (1) takes the following form

$$\begin{align*}
\frac{du_1(t)}{dt} &= (-d_1 + w_1 u_1 u_2)(t) + w_1 M_2 + w_1 c_1 + I_1, \\
\frac{du_2(t)}{dt} &= -d_2 u_2(t) + w_2 l_1 u_1(t) + w_2 M_2 + w_2 c_1 + I_2,
\end{align*}$$

where $l_1, c_1$ are defined as in (11).

Denoting $x(t) = u(t) - u^{(1)}$, we have

$$\begin{align*}
\frac{dx(t)}{dt} &= (-d_1 + w_1 u_1 u_2)x_1(t), \\
\frac{dx_2(t)}{dt} &= -d_2 x_2(t) + w_2 l_1 x_1(t).
\end{align*}$$

It is easy to see that

(i) along the direction $u = (u^{(1)}, u_2)$, the equilibrium $u^{(1)}$ is attractive;

(ii) when $u_1 < u^{(1)}$, then $\frac{dx(t)}{dt} < 0$, so that $u_1(t)$ decreases until $u_1(t_0) \leq p_1$ for some $t_0 > 0$; while $u_2(t)$ remains in $(q_2, +\infty)$ for all $t \geq 0$. Hence, $(u_1(t_0), u_2(t))$ will be attracted to $S_1$;

(iii) similarly, when $u_1 > u^{(1)}$, $(u_1(t), u_2(t))$ will be attracted to $S_2$.

Therefore,

1. $\{(u_1, u_2) : u_1 = u^{(1)}\} \cap S_1$ is in attraction basin of $u^{(1)}$;

2. $\{(u_1, u_2) : u_1 < u^{(1)}\} \cap S_1$ is in attraction basin of $u^{(1)}$;

3. $\{(u_1, u_2) : u_1 > u^{(1)}\} \cap S_1$ is in attraction basin of $u^{(1)}$.

For $u^{(2)}$, $u^{(3)}$, $u^{(4)}$, we can obtain similar conclusions.

In $\Lambda$, consider the dynamics of the following system

$$\frac{dx(t)}{dt} = Ax(t) + \alpha x, \quad (38)$$

where $A$ is defined as (18) and $\alpha = (\tilde{\alpha}_1, \tilde{\alpha}_2)$.
Suppose that $\text{det}(A) > 0$. Then, all eigenvalues of $A$ have positive real-parts, which implies that $u^0$ is unstable. Define $y^T(t) = u^T(t) + A^{-1}\alpha^T$, then

$$\frac{dy^T(t)}{dt} = Ay^T(t). \tag{39}$$

If $\dot{U}(t)$ is its fundamental matrix, then, $y^T(t) = \dot{U}(t)y^T(0)$. Similar to time reversal, define $\tilde{y}^T(t) = \tilde{U}^{-1}(t)\tilde{y}^T(0)$, $\tilde{y}^T(t) = \tilde{y}^T(t) - A^{-1}\alpha^T$. Then, we have

$$\frac{d\tilde{y}^T(t)}{dt} = -A\tilde{y}^T(t), \tag{40}$$

and

$$\frac{d\tilde{U}(t)}{dt} = -A\tilde{U}(t) - \alpha^T, \tag{41}$$

which is an asymptotically stable system and all trajectories converges to $u^A$.

Let $\Gamma_1 = \{(u,t) : (u^2, v) = (\alpha(A^2)^{-1} \alpha t)\}$ be the trajectory of system (41) with initial state $(u^2, v)$, which is also a trajectory of the system (38) with initial state neighbored $u^A$ and passing through $(u^2, v)$ by reversal. Then, the attraction basin of $u^2$ lies in $(\Sigma_1 \cap \{u : u_1 = \frac{a}{\beta}\}) \cup \Gamma_1$.

Similarly define

$$\Gamma_2 = \{(p_1, u^2) : (u^2, v) = (\alpha(A^2)^{-1} \alpha t)\}, \tag{42}$$

and we can prove that

1. the attraction basins of $u^2$ lies in $(\Sigma_2 \cap \{u : u_2 = u_2^2\}) \cup \Gamma_2$;
2. the attraction basins of $u^2$ lies in $(\Sigma_3 \cap \{u : u_2 = u_2^2\}) \cup \Gamma_2$;
3. the attraction basins of $u^2$ lies in $(\Sigma_4 \cap \{u : u_1 = u_1^2\}) \cup \Gamma_2$.

Then, let $\Delta_1$ be the region of $A$ bounded by $\Gamma_1$ and $\Gamma_2$, $\Delta_2$ be the region of $A$ bounded by $\Gamma_1$ and $\Gamma_2$, $\Delta_3$ be the region of $A$ bounded by $\Gamma_2$ and $\Gamma_3$, $\Delta_4$ be the region of $A$ bounded by $\Gamma_3$ and $\Gamma_1$. We claim that the attraction basin of $u^A$

$$A.B(u^A) = \text{int}(S_1 \cup (\Sigma_1 \cap \{u : u_1 < u_1^2\}) \cup (\Sigma_2 \cap \{u : u_2 > u_2^2\}) \cup (\Sigma_3 \cap \{u : u_1 < u_1^2\}) \cup (\Sigma_4 \cap \{u : u_1 = u_1^2\})), \tag{43}$$

where $\text{int}(\cdot)$ denotes the interior of the subset.

Based on previous discussions, we need to prove that any trajectory with initial $u(0) \in \Delta_1$ will converge to $u^A$. Suppose $u(t)$ is an arbitrary solution with initial state in $\Delta_1$. Due to the uniqueness, $u(t)$ does not intersect with boundary of $A.B(u^A)$, which implies that $u(t)$ would stay in $A.B(u^A)$ for all $t \geq 0$. Then, we need to do next is to prove that $u(t)$ cannot be bounded in $\Delta_1$.

If not, $u(t)$ is bounded in $\Delta_1$, and we denote $\omega(u)$ as the $\omega$-limit set of $u(t)$. Then $\omega(u) \neq \emptyset$, and $u^A \not\subset \omega(u)$ obviously. By Poincaré–Bendixon Theorem, we know that $u(t)$ is closed, or $\omega(u)$ is closed. Because every trajectory of system (41) converges to $u^A$ exponentially, so nothing can be said about closed orbit.

Hence, there must exist $t_0 > 0$ such that $u(t_0) \not\subset \Delta_1$. By conditions (5), it would stay in $A.B(u^A) \setminus \Delta$, which implies $u(t)$ converges to $u^A$. Similar conclusions can be derived for $u^2$, $u^3$, $u^4$, and their attraction basins can be described as the interior of subsets

$$S_2 \cup (\Sigma_1 \cap \{u : u_1 > u_1^2\}) \cup (\Sigma_2 \cap \{u : u_2 > u_2^2\}) \cup (\Sigma_3 \cap \{u : u_1 < u_1^2\}) \cup (\Sigma_4 \cap \{u : u_1 > u_1^2\}) \cup \Delta_2,$$

where $\Delta_2$ is a trajectory of the system (41) with initial state $(u^2, v)$, which is also a trajectory of the system (38) with initial state neighbored $u^A$ and passing through $(u^2, v)$ by reversal. Then, the attraction basin of $u^2$ lies in $(\Sigma_1 \cap \{u : u_1 > u_1^2\}) \cup (\Sigma_2 \cap \{u : u_2 > u_2^2\}) \cup (\Sigma_3 \cap \{u : u_1 < u_1^2\}) \cup (\Sigma_4 \cap \{u : u_1 > u_1^2\}) \cup \Delta_3$, respectively.

Summing up, we have the following

**Theorem 4.** Suppose that conditions (5) are satisfied. If $\text{det}(A) > 0$, then, the whole phase plane $R^2$ can be divided into 4 parts, the interior of which are the very attraction basins of equilibria $u^A$, $u^2$, $u^3$, $u^4$, respectively; and the boundaries are attraction basins of $u^2$, $u^3$, $u^4$, respectively.

When $r \geq 1$, $n = 2$, denote $\theta = \frac{m_{k,j} - n_{k,j}}{a_r - \beta_0}$, $\theta = 1, 2, k = 1, \ldots, r$

$$A^{(k, 0)} = \left(\begin{array}{cc}
-d_1 + w_{11}k_1 & w_{12}k_1 \\
-w_{21}k_1 & -d_2 + w_{22}k_1
\end{array}\right).$$

Then, we have

**Theorem 5.** Suppose that (35) (with $n = 2$) is satisfied. If $\text{det}(A^{(k, 0)}) > 0$ are satisfied for all $k, j = 1, \ldots, r$, then, the whole phase plane $R^2$ can be divided into $(r + 1)^2$ parts, the interior of which are the very attraction basins of equilibria $u^A$, $u^2$, $u^3$, $u^4$, respectively; and the boundaries are attraction basins of $u^A$, $u^2$, $u^3$, $u^4$, respectively.

In the following, we discuss the n-neural networks with $r = 1$. Theorem 2 tells us that under conditions (20), there are $3^n$ equilibrium points, $2^n$ of them are stable in subsets $S_1, S_2, S_3$, and $3^n - 2^n$ of them are unstable and located in $\Sigma_1, \Sigma_2 \setminus \Sigma_{2^n-1}, A_1, \ldots, A_{3^n-2^n-2^n-1}$, respectively (See Fig. 4). Next, we are to investigate their corresponding attraction basins. Take a look at dynamics in $\Sigma_1 = \{p_1, q_1\} \times \prod_{i=2}^{\infty}(q_i, \infty)$ for example, in which system (1) reduces to the following equations

$$\begin{align*}
\frac{du_1(t)}{dt} &= (-d_1 + w_{11}l_1)u_1(t) + \sum_{j=2}^{n}w_{1j}M_j + w_{11}c_1 + l_1, \\
\frac{dx_1(t)}{dt} &= -d_1x_1(t) + w_{11}l_1x_1(t) + \sum_{j=2}^{n}w_{1j}M_j + w_{11}c_1 + l_1,
\end{align*}$$

for $i = 2, \ldots, n$, where $l_i, c_i, i = 1, \ldots, n$, are defined as (23). Denote $x(t) = u(t) - u^A$, and we have

$$\begin{align*}
\frac{dx_1(t)}{dt} &= (-d_1 + w_{11}l_1)x_1(t), \\
\frac{dx_1(t)}{dt} &= -d_1x_1(t) + w_{11}l_1x_1(t), \text{ for } i \geq 2.
\end{align*} \tag{46}$$

Then, similar to the analysis on the case with $n = 2$, it is easy to see that, $u^A$ is attractive only along the $n - 1$ dimension surface $x_1 = 0$, i.e., $u_1 = u_1^A$. Correspondingly, $\Sigma_1$ splits into 2 parts, $(u : u_1 < u_1^A) \cap \Sigma_1$ is in attraction basin of $u^A$, $(u : u_1 > u_1^A) \cap \Sigma_1$ is in attraction basin of $u^2$, where $S_1 = (-\infty, p_1) \times \prod_{i=2}^{n}(q_i, \infty), S_2 = \prod_{i=1}^{n}(q_i, \infty)$. Similar conclusions can be derived for $\Sigma_2, \ldots, \Sigma_{2^{n-1}}$. 
Next, consider the dynamics in the subset $A_1 = [p_1, q_1] \times [p_2, q_2] \times \prod_{i=3}^{n}(q_i, +\infty)$, in which system (1) reduces to the following equations
\[
\begin{align*}
\frac{du_1(t)}{dt} &= (d_1 + w_{11}l_1)u_1(t) + w_{12}l_2u_2(t) + \sum_{j=3}^{n} w_{1j}M_j + w_{11}c_1 + w_{12}c_2 + I_1, \\
\frac{du_2(t)}{dt} &= w_{21}l_1u_1(t) + (d_2 + w_{22}l_2)u_2(t) + \sum_{j=3}^{n} w_{2j}M_j + w_{21}c_1 + w_{22}c_2 + I_2, \\
\frac{du_i(t)}{dt} &= -d_iu_i(t) + w_{i1}l_1u_1(t) + w_{i2}l_2u_2(t) + \sum_{j=3}^{n} w_{ij}M_j + w_{i1}c_1 + w_{i2}c_2 + I_i, \quad i \geq 3.
\end{align*}
\]
Denote $x(t) = u(t) - u^{A_1}$, and we have
\[
\begin{align*}
\frac{dx_1(t)}{dt} &= (d_1 + w_{11}l_1)x_1(t) + w_{12}l_2x_2(t), \\
\frac{dx_2(t)}{dt} &= w_{21}l_1x_1(t) + (d_2 + w_{22}l_2)l_2x_2(t), \\
\frac{dx_i(t)}{dt} &= -d_ix_i(t) + w_{i1}l_1x_1(t) + w_{i2}l_2x_2(t), \quad i \geq 3,
\end{align*}
\]
which implies that $u^{A_1}$ is attractive along the surface
\[
\left\{
\begin{array}{l}
(d_1 + w_{11}l_1)x_1 + w_{12}l_2x_2 = 0, \\
w_{21}l_1x_1 + (d_2 + w_{22}l_2)x_2 = 0.
\end{array}
\right.
\] (48)

Suppose all 2-ordered principal minor determinants of $\tilde{A}$ are positive. Then, the above surface reduces to $x_1 = 0, x_2 = 0$, i.e., $u = u^{A_1}, u_2 = u^{A_2}$, which is a n - 2 dimensional surface in subset $A_1$. Compared with $n - 1$ dimension surface, it cannot split $A_1$ directly, which needs further investigation. So are other subsets $A_2, \ldots, A_{n-2, n-2}$.  

5. Discussions

In Sections 2 and 3, we investigate the multistability of neural networks (1) with the activation function given in (2). The proposed method is also applicable when different neurons have different activation functions.

In fact, let $r_1, \ldots, r_n$ be constants such that the activation $f_i$ of the $j$-th neuron has $2r_j$ corner points. Then, we have

**Corollary 1.** Suppose that
\[
\begin{align*}
-d_i(\lambda_j^k + w_{ij}f_i(p_j^k) + \sum_{j \neq i} \max\{w_{ij}m_j^1, w_{ij}m_j^{r+1}\} + I_i &< 0, \\
-d_i(\lambda_j^k + w_{ij}f_i(q_j^k) + \sum_{j \neq i} \min\{w_{ij}m_j^1, w_{ij}m_j^{r+1}\} + I_i &> 0,
\end{align*}
\]
for $k = 1, \ldots, r_i, i, j = 1, 2, \ldots, n$. Then, the dynamical system (1) has $(2r_1 + 1)(2r_2 + 1) \cdots (2r_n + 1)$ equilibrium points in all. Among them, $(r_1 + 1)(r_2 + 1) \cdots (r_n + 1)$ equilibrium points are locally stable and others are unstable.

Similarly, the method can also be used with the neural networks (1) with multilevel sigmoid activation functions as follows:
\[
g_i(x) = \begin{cases} 
-2\tan(\lambda_j x) + \tan(\lambda_j (x + 2r)), & -\infty < x < -2r - 1, \\
2\tan(\lambda_j x) + \tan(\lambda_j (x - 2r)), & 2r - 1 \leq x \leq 2r + 1, \\
2\tan(\lambda_j x) + \tan(\lambda_j (x - 2r)), & 2r + 1 < x < \infty,
\end{cases}
\] (50)
where $\lambda_j > 0, j = 1, \ldots, n, k = -r, \ldots, -1, 0, 1, \ldots, r$. And we have

**Corollary 2.** Suppose that there are constants $p_j^k, q_j^k \in (2k - 1 + 2k + 1), k = -r, \ldots, -1, 0, 1, \ldots, r$ such that
\[
-\infty < p_j^k < q_j^k < p_j^{r+1} < q_j^{r+1} < \cdots < p_j^r < q_j^r < +\infty,
\]
and
\[
\begin{align*}
-d_i(\lambda_j^k + w_{ij}g_i(p_j^k) + \sum_{j \neq i} (2\tanh(\lambda_j) + 1)|w_{ij}| + I_i &< 0, \\
-d_i(\lambda_j^k + w_{ij}g_i(q_j^k) - \sum_{j \neq i} (2\tanh(\lambda_j) + 1)|w_{ij}| + I_i &> 0,
\end{align*}
\]
for $i, j = 1, 2, \ldots, n, k = -r, \ldots, -1, 0, 1, \ldots, r$. Then, the dynamical system (1) has $(4r + 3)^n$ equilibrium points. Among them, $(2r + 2)^n$ are locally stable.

It should be noted that under conditions (5), (20) and (35), the pattern storage of neural network (1) with activation function (2) reaches its maximum. And in this case, the conditions $w_{ij}I_i > d_i, i = 1, \ldots, n$, are necessary.

In fact, for any fixed index $i$, there exists at least one equilibrium point $u^*$ in the subset $\prod_{j \neq i}(-\infty, p_j) \times (-\infty, p_i) \times \prod_{j \neq i}(q_j, \infty)$, and at least one equilibrium point $v^*$ in the subset $\prod_{j \neq i}(-\infty, p_j) \times (q_i, \infty) \times \prod_{j \neq i}(q_j, \infty)$, where the index sets $N_1, N_2$ satisfy that $N_1 \cup N_2 \cup \{i\} = \{1, \ldots, n\}, N_1 \cap N_2 = \emptyset$. The $i$-th component of $u^*$ can be written as
\[
u_i^* = \frac{w_{ij}m_i + \sum_{j \in N_1} w_{ij}m_j + \sum_{j \in N_2} w_{ij}M_j + I_i}{d_i} < p_i,
\] (52)
while
\[
u_i^* = \frac{w_{ij}M_i + \sum_{j \in N_1} w_{ij}m_j + \sum_{j \in N_2} w_{ij}M_j + I_i}{d_i} > q_i.
\] (53)
Then, we have
\[
u_i^* - u_i^* = \frac{w_{ij}M_i - w_{ij}m_i}{d_i} = \frac{w_{ij}(q_i - p_i)}{d_i} > q_i - p_i,
\] (54)
which implies that $w_{ij}I_i > d_i$.

It is natural to ask what will happen when $w_{ij}I_i \leq d_i$, in particular, when $w_{ij} < 0$, i.e., the $i$-th neuron is self-inhibited. In this case, we find that system (1) may have more than one equilibrium point, but the number must less than $3^r$ or $(2r + 1)^n$. We will not discuss it in detail here. And just explain it by an example.

Consider the following 2-neuron neural network:
\[
\begin{align*}
\frac{du_1(t)}{dt} &= -2u_1(t) + 5f_1(u_1(t)) - f_2(u_2(t)) + 1, \\
\frac{du_2(t)}{dt} &= -4u_2(t) - f_1(u_1(t)) + f_2(u_2(t)) - 1,
\end{align*}
\]
(55)
where the activation function is $f_i(x) = \frac{|x^{i+1}| - |x^{i-1}|}{|x|}, i = 1, 2$. It has 3 equilibrium points $(-2, 0), (-0.4, -0.2), (10/3, -2/3)$ in all. And the dynamics with more than 175 initial states are illustrated in Figs. 6 and 7.

6. Simulations

In the following, we present three more examples to illustrate the effectiveness of the theoretical results.
Example 1. Consider the following neural network with 2-neurons:

\[
\begin{align*}
\frac{du_1(t)}{dt} &= -u_1(t) + 4f_1(u_1(t)) + f_2(u_2(t)) - 1, \\
\frac{du_2(t)}{dt} &= -2u_2(t) + f_1(u_1(t)) + 5f_2(u_2(t)) - 1,
\end{align*}
\]

where the activation functions are \( f_i(x) = \frac{|x+1|-|x-1|}{2} \), \( i = 1, 2 \).

It is easy to see that the conditions (5) are satisfied. Therefore, by Theorem 1, there exist 9 equilibria, and 4 of them are locally stable while others are unstable. In fact, the equilibrium points are \((-6, -3.5), (-13/3, 2/3), (-4, 1.5), (2/3, -16/3), (1/4, 1/4), (0, 2), (2, -2.5), (3, 0), (4, 2.5)\).

And

\[
\begin{align*}
\Gamma_1 : & \quad \begin{cases} 
\frac{du_1(t)}{dt} = -\frac{1}{2}e^{-2t} + \frac{1}{4}e^{-t} + \frac{1}{4}, \\
\frac{du_2(t)}{dt} = \frac{1}{2}e^{-2t} + \frac{1}{4}e^{-t} + \frac{1}{4}; 
\end{cases} \\
\Gamma_2 : & \quad \begin{cases} 
\frac{du_1(t)}{dt} = \frac{5}{6}e^{-2t} - \frac{5}{12}e^{-t} + \frac{1}{4}, \\
\frac{du_2(t)}{dt} = \frac{5}{6}e^{-2t} - \frac{5}{12}e^{-t} + \frac{1}{4}; 
\end{cases} \\
\Gamma_3 : & \quad \begin{cases} 
\frac{du_1(t)}{dt} = \frac{1}{2}e^{-2t} + \frac{1}{4}e^{-t} + \frac{1}{4}, \\
\frac{du_2(t)}{dt} = -\frac{1}{2}e^{-2t} + \frac{1}{4}e^{-t} + \frac{1}{4}; 
\end{cases}
\end{align*}
\]

The dynamics of system (56) are illustrated in Figs. 8 and 9, where evolutions of more than 220 initial states have been tracked. It shows that 4 equilibrium points located in \( S_1, S_2, S_3, S_4 \) are stable, other 4 equilibrium points located in \( S_1, S_2, S_3, S_4 \) are stable in

\[
\begin{align*}
\{u \in S_1 : u_1 = 0\} \cup \Gamma_1, \quad & \{u \in S_2 : u_2 = 2/3\} \cup \Gamma_2, \\
\{u \in S_3 : u_1 = 2/3\} \cup \Gamma_3, \quad & \{u \in S_4 : u_2 = 0\} \cup \Gamma_4,
\end{align*}
\]

respectively, and 1 equilibrium point located in \( \Lambda \) is unstable, as confirmed by theoretical derivations. In the following figures, solutions with 184 initial states in attraction basins of \( u^{31}, u^{32}, u^{33}, u^{34} \) are depicted; and other 40 solutions with initial states \( u_1 = 0 \), or \( u_1 = 2/3 \), or \( u_2 = 0 \), or \( u_2 = 2/3 \), respectively, are depicted by straight lines. The 4 curves represent \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \) in \( \Lambda \), respectively.
Consider the following 3-neuron neural network:

\[
\begin{align*}
\frac{du_1(t)}{dt} &= -2u_1(t) + 6f(u_1(t)) - \frac{1}{2} f(u_2(t)) - 1, \\
\frac{du_2(t)}{dt} &= -\frac{3}{2} u_2(t) - \frac{1}{2} f(u_1(t)) + 4f(u_2(t)) - \frac{1}{2},
\end{align*}
\]

(57)

where the activation function is

\[
f(x) = \begin{cases} 
-1 & x \leq -1, \\
1 & 1 < x \leq 3, \\
3 & 3 < x \leq 5, \\
5 & x > 5.
\end{cases}
\]

It is easy to see that the conditions (35) are satisfied. By Theorem 3, there are 5² equilibrium points located in S₁, ..., S₅, 12 equilibrium points located in S₁, ..., S₁₂ are stable, and 3 of them are locally stable while others are unstable.

The dynamics of system (57) are illustrated in Figs. 10 and 11, where evolutions of more than 500 initial states have been tracked. It shows that 9 equilibrium points located in S₁, ..., S₉ are stable, 12 equilibrium points located in S₁, ..., S₁₂ are stable in the sets

\[
\begin{align*}
[u \in S_1 : u_1 &= 0.625], & [u \in S_2 : u_1 &= 3.625], \\
[u \in S_3 : u_2 &= 3.2], & [u \in S_4 : u_2 &= 3.6], \\
[u \in S_5 : u_2 &= 4], & [u \in S_6 : u_1 &= 0.375], \\
[u \in S_7 : u_1 &= 3.375], & [u \in S_8 : u_2 &= 0], \\
[u \in S_9 : u_2 &= 0.4], & [u \in S_{10} : u_2 &= 0.8], \\
[u \in S_{11} : u_1 &= 0.125], & [u \in S_{12} : u_1 &= 3.125],
\end{align*}
\]

respectively, and 4 equilibrium points located in A₁, ..., A₄ respectively are unstable. These figures confirm the theoretical results. In these figures, solutions with initial states in attraction basins of u₁, ..., u₅ are depicted, solutions with initial states u₁ = 0.625, 3.625, 0.375, 3.375, 0.125, 3.125 or u₂ = 3.2, 3.6, 4, 0.4, 0.8, respectively, are depicted by straight lines. And the 16 curves represent Γ₁, ..., Γ₁₆ in A₁, ..., A₄, respectively.

Example 2. Consider the following 2-neuron neural network:

\[
\begin{align*}
\frac{du_1(t)}{dt} &= -u_1(t) + 5f(u_1(t)) - f(u_2(t)) - f(u_1(t)) - 1, \\
\frac{du_2(t)}{dt} &= -2u_2(t) - f(u_1(t)) + 6f(u_2(t)) - f(u_3(t)) + 1, \quad (59)
\end{align*}
\]

with activation function \( f(x) = \frac{|x + 1| - |x - 1|}{4}. \)

It is easy to see that the conditions (20) are satisfied. Therefore, by Theorem 2, system (59) has 3³ equilibrium points, 3 of them are locally stable while others are unstable. In this paper, we will not discuss the complete attraction basin for each equilibrium point, because it is quite complicated and we leave it as an issue for later investigation.

Here, we just show the dynamics in part of attraction basins of u₁, ..., u₈ in Figs. 12–15, where evolutions of more than 350 initial states have been tracked. It can be seen that 8 equilibrium points located in S₁, S₂, S₃, S₄, S₅, S₆, S₇, S₈ are locally stable, 12 equilibrium points located in S₁, ..., S₁₂ are stable in

\[
\begin{align*}
[u \in S_1 : u_1 &= 0.75], & [u \in S_2 : u_2 &= -0.25], \\
[u \in S_3 : u_2 &= 0.25], & [u \in S_4 : u_1 &= 0.25], \\
[u \in S_5 : u_3 &= -0.4], & [u \in S_6 : u_3 &= -0.4], \\
[u \in S_7 : u_1 &= 0.25], & [u \in S_8 : u_1 &= 0.25], \\
[u \in S_{10} : u_2 &= -0.75], & [u \in S_{11} : u_2 &= -0.5], \\
[u \in S_{12} : u_1 &= -0.25],
\end{align*}
\]

respectively, and 7 equilibrium points located in A₁, ..., A₇ respectively are unstable, which confirm the theoretical results.

7. Conclusions

In this paper, we study the neural networks with a class of activation functions, which are nondecreasing piecewise linear with 2r(r ≥ 1) corner points. We prove that such neural networks have multiple equilibria. Some of them are locally stable and others
are unstable. More precisely, such neural networks have \((2r+1)^n\) equilibria in all, \((r+1)^n\) of which are locally exponentially stable and others are unstable. We also give the attraction region for each locally stable equilibrium. For the 2-neuron neural networks, the attraction basins of stable equilibrium points can be precisely depicted.

**Acknowledgements**

Wang Lili is supported by Graduate Innovation Foundation of Fudan University under Grant EYH1411028. Lu Wenlian is supported by the National Natural Sciences Foundation of China under Grant No. 60804044, and sponsored by Shanghai Pujiang Program No. 08PJ14019. Chen Tiaping is supported by the National Natural Sciences Foundation of China under Grant No. 60774074, 60974015 and SGST09DZ2272900.

**References**


