Synchronization of identical neural networks and other systems with an adaptive coupling strength‡

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SUMMARY

In this paper, a new scheme to synchronize linearly or nonlinearly coupled identical circuit systems, which include neural networks and other systems, with an adaptive coupling strength is proposed. Unlike other adaptive schemes that synchronize coupled circuit systems to a specified trajectory (or an equilibrium point) of the uncoupled node by adding negative feedbacks adaptively, here the new adaptive scheme for the coupling strength is used to synchronize coupled systems without knowing the final synchronization trajectory. Moreover, the adaptive scheme is applicable when the coupling matrix is unknown or time-varying. The validity of the new adaptive scheme is also proved rigorously. Finally, several numerical simulations to synchronize coupled neural networks, Chua’s circuits and Lorenz systems, are also given to show the effectiveness of the theory. Copyright © 2009 John Wiley & Sons, Ltd.

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KEY WORDS: synchronization; an adaptive coupling strength; nonlinearly coupling; unknown or time-varying coupling matrix; neural networks

1. INTRODUCTION

Recently, an increasing interest has been devoted to the study of complex networks, see [1–6], which can be regarded as a composition and interaction of several dynamical nodes. Among them, identical synchronization of coupled complex networks has attracted more attention, since

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synchronization can not only explain many natural phenomena [7], but also have many applications, such as neural networks, image processing, secure communication, etc., see [8–14].

Generally, linearly coupled circuit systems can be described as

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j \neq i} a_{ij} \Gamma[x_j(t) - x_i(t)], \quad i = 1, 2, \ldots, N$$

(1)

where $x_i(t) = (x_1^i(t), \ldots, x_n^i(t))^T \in \mathbb{R}^n$; $f(\cdot, t): \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is continuous. The outer coupling matrix $A = (a_{ij})$ satisfies $a_{ij} \geq 0$, for $i \neq j$, and let $a_{ii} = -\sum_{j=1, j \neq i}^{N} a_{ij}$, while the inner coupling matrix $\Gamma = \text{diag}\{\gamma_1, \ldots, \gamma_n\}$ is non-negative definite and parameter $c$ denotes the coupling strength.

If $\lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0$ for all $i, j = 1, 2, \ldots, N$, where $\|\cdot\|$ denotes some norm, then coupled systems (1) are said to be identically (or completely) synchronized. In the following, we will investigate this identical (or complete) synchronization phenomenon.

Hitherto, many approaches and criteria to ensure complete synchronization have been derived, see [15–26]. For example, in [16], the authors found that certain subsystems of nonlinear, chaotic systems can be made to synchronize by linking them with common signal, which may have the application on the neural processes; in [17, 18] the authors presented a master stability function based on the transverse Lyapunov exponents to study local synchronization; in [19, 20], a distance from synchronization manifold to each state was defined to study the global synchronization; in [21, 22], the left eigenvector corresponding to the zero eigenvalue of the diffusive coupling matrix is utilized to investigate the global synchronization; Grassi and Mascolo [23] applied the concept of observer from system theory to synchronizing high-order oscillators; Wu et al. [24] investigated chaos synchronization of the master–slave Chua’s circuits by a general linear state error feedback controller with propagation delay; Jalili et al. [25] investigated the synchronization of dynamical networks by using the connection graph stability method, which regarded the network topology as a graph; and Szatmári and Chua [26] studied synchronization mechanism among cells in reaction–diffusion systems, showed the similarities to basic pulse synchronization technique, and presented that the passive coupling among completely stable cells might produce very interesting dynamical behavior.

In [20], the authors investigate linearly coupled recurrently connected neural network (including Hopfield neural networks, cellular neural networks, and others) with delays and prove that under some mild conditions, if the coupling strength is large enough, then synchronization can be realized. On the other hand, it is also pointed out by simulations that the theoretical strength to ensure synchronization is much bigger than the strength needed in practice. Therefore, how to find smaller strength is significant in practice, and this point is one of the main purposes of this paper.

Besides linearly coupled systems, in some cases, we need to investigate the nonlinearly coupled systems. Because in practice, state variables $x_i(t)$ may be unobservable; what we can observe is $g(x_i(t))$, which is a nonlinear function of the state. How to synchronize coupled systems with the observed data is of great significance, and in [27], the following nonlinearly coupled circuit systems

$$\dot{x}_i(t) = f(x_i(t), t) + c \sum_{j \neq i} a_{ij}[g(x_j(t)) - g(x_i(t))], \quad i = 1, 2, \ldots, N$$

(2)
were investigated, where \( g(x_j(t)) = (g_1(x_1^T(t)), \ldots, g_n(x_n^T(t))^T \) and every \( g_i(\cdot) \) is a nonlinear monotone increasing function.

However, in many cases, the coupling matrix \( A = (a_{ij}) \) may be unknown or time-varying, so the other main purpose of this paper is to find a useful scheme to synchronize such coupled chaotic systems.

To realize the above two purposes, we apply the adaptive approach to the coupling strength \( c \). In this case, coupled systems (1) and (2) can be replaced by

\[
\dot{x}_i(t) = f(x_i(t), t) + c(t) \sum_{j \neq i} a_{ij} [x_j(t) - x_i(t)], \quad i = 1, 2, \ldots, N
\]

(3)

\[
\dot{x}_i(t) = f(x_i(t), t) + c(t) \sum_{j \neq i} a_{ij} [g(x_j(t)) - g(x_i(t))], \quad i = 1, 2, \ldots, N
\]

(4)

In fact, the adaptation of parameters has been widely used in the signal processing and other research fields. Recently, the adaptive approach is used to synchronize master–slave systems or mutually coupled systems to a specified trajectory of the uncoupled node by adding negative feedbacks (see [28–36]). For example, in [28–30], the authors investigated how to synchronize the coupled systems

\[
\dot{x}_i(t) = f(x_i(t), t) + h_i(x_1(t), x_2(t), \ldots, x_N(t)), \quad 1 \leq i \leq N
\]

(5)

to a specified trajectory \( \dot{s}(t) = f(s(t), t) \) by adding negative feedback controls \(-d_i(t)[x_i(t) - s(t)]\) with the adaptation rule \( \dot{d}_i = k_i \| x_i(t) - s(t) \|_2^2 \).

However, the above adaptation rule cannot realize the synchronization for models (1) and (2), because the final synchronization trajectory, which generally is not a trajectory of the uncoupled system, is unknown.

In this paper, we propose a simple scheme to synchronize chaotic systems with an adaptive coupling strength described by Equations (3) and (4). Its validity is proved rigorously. Moreover, simulations are also given to verify the effectiveness of the synchronization scheme.

The rest of this paper is organized as follows. In Section 2, some necessary definitions, lemmas, and hypotheses are given, then a new adaptive synchronization scheme is designed and two theorems are given to prove its validity rigorously for coupled networks with an unknown or time-varying coupling matrix; in Section 3, several simulations are given to verify the theoretical results; and the paper is concluded in Section 4.

### 2. SYNCHRONIZATION WITH AN ADAPTIVE COUPLING STRENGTH

In this section, we make some preparations. Some definitions, denotations, and lemmas throughout the paper are presented.

**Definition 1**

The set \( S = \{(x_1(t))^T, x_2(t))^T, \ldots, x_N(t)^T \}: x_i(t) = x_j(t); i, j = 1, 2, \ldots, N \} \) is called the synchronization manifold.
If a matrix \( A \) is definite, then for any two vectors \( X_1, X_2 \) we always assume that

\[
\sum_{i=1}^{N} x_i a_{ii} \geq 0
\]

and an \( N \times N \) matrix \( A \) is said:

1. \( a_{ij} \geq 0, i \neq j, a_{ii} = -\sum_{j=1, j \neq i}^{N} a_{ij}, i = 1, 2, \ldots, N \)
2. \( A \) is irreducible.

Furthermore, if \( A \in \mathcal{A}_1 \) and \( a_{ij} = a_{ji}, i \neq j \), then we say \( A \in \mathcal{A}_2 \).

**Lemma 1**

If a matrix \( A_{N \times N} \in \mathcal{A}_1 \), then (see [21, 22])

1. \( 1 = (1, 1, \ldots, 1)^T \) is the right eigenvector of \( A \) corresponding to eigenvalue 0 with multiplicity 1;

2. The left eigenvector \( \xi = (\xi_1, \xi_2, \ldots, \xi_N)^T \in \mathbb{R}^N \) of matrix \( A \) corresponding to the eigenvalue 0 has the following property: \( \xi_i > 0, i = 1, 2, \ldots, N \). Without loss of generality, in the following, we always assume that \( \sum_{i=1}^{N} \xi_i = 1 \).

**Lemma 2**

For any matrix \( A_{N \times N} \in \mathcal{A}_2 \), its eigenvalues are all real and can be sorted as

\[
0 = \lambda_1(A) > \lambda_2(A) \geq \lambda_3(A) \geq \ldots \geq \lambda_N(A)
\]

**Lemma 3 (I. Barbălat, see [37])**

If \( \phi \) is a real function of the real variable \( t \), defined and uniformly continuous for \( t > 0 \) and if the limit of the integral \( \int_{0}^{t} \phi(\tau) d\tau \) as \( t \) tends to infinity exists and is a finite number, then

\[
\lim_{t \to +\infty} \phi(t) = 0
\]

**Lemma 4**

Suppose an \( N \times N \) matrix \( \tilde{A} = (\tilde{a}_{ij}) \in \mathcal{A}_2 \) and an \( n \times n \) matrix \( \Gamma \) is symmetric and semi-positive definite, then for any two vectors \( X = (x_1^T, \ldots, x_N^T)^T \) and \( Y = (y_1^T, \ldots, y_N^T)^T \), where \( x_i, y_i \in \mathbb{R}^n, i = 1, \ldots, N \), we have (see [19])

\[
X^T(\tilde{A} \otimes \Gamma)Y = \sum_{i > j} \tilde{a}_{ij} (x_i - x_j)^T \Gamma (y_i - y_j)
\]

where \( \otimes \) is the Kronecker product.

### 2.1. Synchronization with a unknown constant coupling matrix

In this subsection, we consider the following linearly coupled systems with an adaptive coupling strength:

\[
\dot{x}_i(t) = f(x_i(t), t) + c(t) \sum_{j \neq i} a_{ij} [x_j(t) - x_i(t)]
\]

where the coupling matrix \( A = (a_{ij})_{i,j=1}^{N} \in \mathcal{A}_1 \) can be asymmetric and unknown.

Denote \( X(t) = (x_1^T(t), \ldots, x_N^T(t))^T \), \( F(X(t), t) = (f(x_1(t), t)^T, \ldots, f(x_N(t), t)^T)^T \), \( A_\Gamma = A \otimes \Gamma \). Then coupled systems (8) can be rewritten in a compact form as

\[
\dot{X}(t) = F(X(t), t) + c(t) A_\Gamma X(t)
\]
Theorem 1
Assume that the constant coupling matrix \( A \in \mathbb{A}_1 \) and the function \( f(x,t) \) is continuous on \( (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \) and locally uniformly bounded on \( x \) with respect to \( t \), and \( f \in \text{QUAD}(P, \Delta, \sigma) \), i.e. there exists a constant \( \sigma > 0 \), a positive definite diagonal matrix \( P = \text{diag}\{p_1, \ldots, p_n\} \), and a nonnegative definite diagonal matrix \( \Delta = \text{diag}\{\delta_1, \ldots, \delta_n\} \), such that
\[
(x-y)^T P [f(x,t) - f(y,t) - \Delta(x-y)] \leq -\sigma (x-y)^T (x-y)
\] (10)
holds for any \( x, y \in \mathbb{R}^n \); especially, matrix \( \Delta \) satisfies for \( j = 1, \ldots, n \), if \( \gamma_j = 0 \), then \( \delta_j = 0 \). Moreover, assume that for any initial values \( x_i(0), i = 1, \ldots, N \), there exists at least one index \( j_0 \), such that the trajectory \( x_{j_0}(t) \) is bounded for any time \( t \). Therefore, the following coupled circuit systems with an adaptive coupling strength
\[
\dot{X}(t) = F(X(t), t) + c(t)\Delta\Gamma X(t)
\] (11)
\[
\dot{c}(t) = -\frac{\alpha}{2} (X(t)^T \tilde{\Delta} X(t))
\]
can achieve synchronization, where \( \alpha > 0 \), \( c(0) = 0 \), \( \tilde{\Delta} = \tilde{\Delta} \otimes \Gamma \), and \( \tilde{\Delta} \in \mathbb{R}^{N \times N} \) can be any matrix belonging to \( \mathbb{A}_2 \); moreover, as time reaches infinity, the coupling strength \( c(t) \) will finally converge to a constant \( c^* \), which is determined by the initial value \( c(0) \) and the parameter \( \alpha \).

Proof
Define a reference trajectory in the synchronization manifold by \( X_\xi(t) = (x_\xi(t)^T, \ldots, x_\xi(t)^T)^T \in \mathcal{S} \), where \( x_\xi(t) = \sum_{i=1}^N \xi_i x_i(t) \), and define \( \tilde{X}(t) = (\tilde{x}_1(t)^T, \ldots, \tilde{x}_N(t)^T)^T \), where \( \tilde{x}_i(t) = x_i(t) - x_\xi(t), i = 1, \ldots, N \). Then, the identical synchronization of coupled networks (11) is equivalent to prove: \( \lim_{t \to +\infty} \tilde{X}(t) = 0 \).

Moreover, denote \( \Xi = \text{diag}\{\xi_1, \ldots, \xi_N\}, \ U = \Xi - \xi_\xi^T, \ F(X_\xi(t), t) = (f(x_\xi(t), t)^T, \ldots, f(x_\xi(t), t)^T)^T. \ \Delta = I_N \otimes \Delta, \ \Xi = \Xi \otimes P, \) and \( \ U = U \otimes P \), where \( I_N \) is the identity matrix and \( \otimes \) is the Kronecker product.

It is easy to check directly that
\[
-U \in \mathbb{A}_2, \quad (\Xi A)^T = \frac{1}{2}(\Xi A + A^T \Xi) \in \mathbb{A}_2
\] (12)
\[
\Xi \Delta = \Upsilon \Delta \quad \text{and} \quad \Xi \tilde{X} = U \tilde{X}
\] (13)
then by the QUAD inequality (10), we have
\[
\tilde{X}(t)^T \Xi (F(X(t), t) - \Delta \tilde{X}(t)) = \tilde{X}(t)^T \Xi (F(X(t), t) - F(X_\xi(t), t) - \Delta \tilde{X}(t))
\]
\[
\leq -\sigma \max_k p_k \tilde{X}(t)^T \Xi \tilde{X}(t)
\] (14)

For \( j = 1, \ldots, n \), if \( \gamma_j > 0 \), then by Lemma 2, we can pick a sufficiently small constant \( \eta > 0 \) such that
\[
\gamma_j (p_j \dot{\xi}_2(\Xi A)^T) - \eta \dot{\xi}_N (\tilde{\Delta}) \leq 0
\] (15)
and for this chosen \( \eta \), pick a sufficiently large constant \( L > 0 \) such that
\[
\eta L \gamma_j \dot{\xi}_2(\tilde{\Delta}) - \sigma \dot{\xi}_N (U) \leq 0
\] (16)
On the other hand, if $\gamma_j = 0$, then with the assumption for matrix $\Delta$ in the Theorem, we take $\delta_j = 0$, and it is clear that inequalities (15) and (16) also hold.

With these parameters, define a candidate function

$$V(t) = \frac{1}{2} \ddot{X}(t)^T \Xi \ddot{X}(t) + \frac{\eta}{\alpha} (L - c(t))^2$$

$$= \frac{1}{2} \sum_{i=1}^{N} \zeta_i(x_i(t) - x_i(t))^T P(x_i(t) - x_i(t)) + \frac{\eta}{\alpha} (L - c(t))^2$$

(17)

where $\eta$ and $L$ are defined such that inequalities (15) and (16) hold.

Differentiating it and combining with the inequality (14), we have

$$\dot{V}(t) = \ddot{X}(t)^T \Xi [F(X(t), t) + c(t)A_F \dot{X}(t)] + \eta(L - c(t))\ddot{X}(t)$$

$$= \ddot{X}(t)^T \Xi [F(X(t), t) - \Delta \ddot{X}(t)] + c(t)\ddot{X}(t) + (\Xi A_F)^s - \eta \ddot{X}(t)$$

$$+ \ddot{X}(t)(U\Delta + \eta \dot{A}) \ddot{X}(t)$$

$$\leq - \frac{c_0}{\max_k p_k} \ddot{X}(t)^T \Xi \ddot{X}(t) \leq 0$$

(18)

which implies

$$\int_0^t \ddot{X}(s)^T \Xi \ddot{X}(s) \, ds \leq \frac{\max_k p_k}{\sigma} (V(0) - V(t)) \leq \frac{\max_k p_k}{\sigma} V(0) \leq +\infty$$

(19)

Additionally, by the facts $\dot{V}(t) \leq 0$, $\dot{c}(t) \geq 0$, and

$$0 \leq \frac{\eta}{\alpha} (L - c(t))^2 \leq V(t) \leq V(0)$$

(20)

we conclude that $c(t)$ and $\ddot{x}(t) = x_i(t) - x_i(t)$ are all bounded; moreover, from the definition of $x_i(t)$, we can also get that $x_i(t) - x_i(t), i, j = 1, \ldots, N$, are all bounded. Owing to the assumption that $x_{j_0}(t)$ is bounded, we can also conclude that $x_i(t), i = 1, \ldots, N$, are all bounded. Note that for $i = 1, \ldots, N$,

$$\dot{x}_i(t) = \sum_{k=1}^{N} \zeta_k[f(x_i(t), t) - f(x_k(t), t)] + c(t) \sum_{j=1}^{N} a_{ij} \Gamma[x_j(t) - x_i(t)]$$

implies that the derivatives $\dot{x}_i(t), i = 1, \ldots, N$, are bounded. Therefore, $\ddot{x}_i(t)$ are all uniformly continuous; namely, $\ddot{X}(t)^T \Xi \dd dot{X}(t) = \sum_{i=1}^{N} \zeta_i \ddot{x}_i(t) \ddot{x}_i(t)$ is uniformly continuous. Combining with
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(19) and the Barbálat Lemma, we have \( \lim_{t \to +\infty} \tilde{X}(t)^T \Xi \tilde{X}(t) = 0 \), i.e. \( \lim_{t \to +\infty} \tilde{X}(t) = 0 \)

therefore, synchronization can be realized, \( \lim_{t \to +\infty} x_i(t) - x_j(t) = 0 \), \( i, j = 1, \ldots, N \).

Finally, recall the Lemma 4, the inequality (19) also implies that the trajectories

\[
c(t) = -\frac{\alpha}{2} \int_0^t X(s)^T \tilde{A} X(s) \, ds = -\frac{\alpha}{2} \int_0^t \sum_{i<j} \tilde{a}_{ij}(x_i(s) - x_j(s))^T \Gamma(x_i(s) - x_j(s)) \, ds
\]

is a Cauchy series as \( t \to +\infty \) and converges to some constant. Theorem 1 is proved completely.

Remark 1

The QUAD condition (10) implies that the term \( \Delta x \) is a linear state feedback that globally stabilizes the system \( \dot{x} = f(x(t), t) \), and this property is also known as quadratically stabilizable in the control literature. It can be shown that the function \( f \in \text{QUAD}(P, \Delta, \sigma) \) holds for several well-known chaotic oscillators, such as Chua’s circuit, the Rössler-like system, the Lorenz system, and so on. Similar definitions have been widely adopted in the synchronization literature, see [21, 22].

Remark 2

Of course, if a function \( f \) satisfies the Lipschitz condition, then the QUAD assumption obviously holds. In other words, this QUAD assumption for function \( f \) is less conservative than the Lipschitz condition. In fact, the Lipschitz condition requires the computation of all components of a vector, so by synchronizing the network, one needs to add the coupling to all components, i.e. \( \Gamma \) should be symmetric and positive definite; while the QUAD condition only requires the computation of part of the components of a vector, so by synchronizing the network, one only needs to add the coupling to these components, i.e. \( \Gamma \) can be a symmetric and semi-positive matrix, which will greatly reduce the complexity of computation especially for large-scale networks.

Remark 3

Intuitually, larger \( c(0) \) and \( \alpha \) can make coupled systems synchronize faster (which means shorter synchronization time), and the final constant \( c^* \) may also be larger (which means more cost for synchronization). So the first advantage of this adaptive method is that one can choose suitable parameters \( c(0) \) and \( \alpha \) in order to get a balance between the synchronization cost and the synchronization time. The second advantage of this adaptive method is that it can be applied on the case that the system parameters are unknown. The third advantage of this adaptive method is that it can be used to track control of some circuit systems or complex networks. In fact, the above condition for matrix \( A \) can be easily generalized to the case that \( A \) has a spanning tree, i.e. the network is master–slave (leader–follower) coupled; for this control problem we refer the interested readers to [22, 38, 39].

Remark 4

According to the above theorem, we know that the matrix \( \tilde{A} \) can be any matrix belonging to class \( A_2 \), which means that, from the graph theory, the adaptive rule can be obtained from any connected graph. This flexibility of \( \tilde{A} \) makes our adaptive scheme more general than many existed results, such as [38–40].
2.2. Synchronization with a time-varying coupling matrix

In this subsection, we consider the following time-varying coupled systems with an adaptive coupling strength:

$$\dot{x}_i(t) = f(x_i(t), t) + c(t) \sum_{j \neq i} a_{ij}(t)[g(x_j(t)) - g(x_i(t))]$$  \hspace{1cm} (21)

where $g(x_i(t)) = (g_1(x_i^1(t)), \ldots, g_n(x_i^n(t)))^T$.

Denote $G(X(t)) = (g^T(x_1(t)), \ldots, g^T(x_N(t)))^T$ and $A(t) = A(t) \otimes I_n$, then we have

$$\dot{X}(t) = F(X(t), t) + c(t)A(t)G(X(t))$$  \hspace{1cm} (22)

**Theorem 2**

Without loss of generality, we assume $(g_k(u) - g_k(v))/(u - v) \geq 1$ for any scalars $u \neq v$, when $k = 1, \ldots, m$ and $g_k = 0$ for $k = m + 1, \ldots, n$. Moreover, we assume that the function $f(x, t)$ is continuous on $(x, t) \in R^n \times R^+$ and locally uniformly bounded on $x$ with respect to $t$, and $f \in QUAD(P, \Delta, \eta)$, which is defined by inequality (10), and matrix $\Delta$ satisfies $\delta_j > 0$ when $j = 1, \ldots, m$; while $\delta_j = 0$ when $j = m + 1, \ldots, n$. Another assumption is that for any initial values $x_i(0), i = 1, \ldots, N$, there exists at least one index $j_0$, such that the trajectory $x_{j_0}(t)$ is bounded for any time $t$. If there exists a scalar $\lambda^*$, such that for all $t \geq 0$, $A(t) \in \mathbf{A}^2$ and $\dot{\lambda}_2(A(t)) \leq \lambda^* < 0$; then the following time-varying coupled systems with an adaptive coupling strength $c(t)$:

$$\dot{X}(t) = F(X(t), t) + c(t)A(t)G(X(t))$$

$$\dot{c}(t) = -\frac{\alpha}{2}X(t)^T \tilde{\Lambda}X(t)$$  \hspace{1cm} (23)

can achieve identical synchronization, where $\alpha > 0$, $c(0) = 0$, $\tilde{\Lambda} = \tilde{\Lambda} \otimes \tilde{\Gamma}$, $\tilde{\Lambda} \in \mathbf{A}^2$, and $\tilde{\Gamma} = \text{diag}(1, \ldots, m, 0, \ldots, 0) \in R^{n \times n}$; moreover, as time reaches infinity, the coupling strength $c(t)$ will finally converge to a constant $c^*$, which is determined by the initial value $c(0)$ and the parameter $\alpha$.

**Proof**

Since $A(t) \in \mathbf{A}^2$, hence for all $t > 0$, $\xi = 1^T/N$ is the left eigenvector corresponding to eigenvalue 0 of $A(t)$.

Using the same function

$$V(t) = \frac{1}{2} \dot{X}(t)^T \Xi \dot{X}(t) + \frac{\eta}{\alpha} (L - c(t))^2$$

where $\Xi = \Xi \otimes P$. Differentiating it, we have

$$\dot{V}(t) = \dot{X}(t)^T \Xi [F(X(t), t) + c(t)A(t)G(X(t))] + \eta (L - c(t)) \dot{X}(t)^T \tilde{\Lambda} \ddot{X}(t)$$

$$= \dot{X}(t)^T \Xi [F(X(t), t) - \Delta \ddot{X}(t)] + \dot{X}(t)^T (U \Delta + \eta \tilde{\Lambda}) \ddot{X}(t)$$

$$+ c(t) \dot{X}(t)^T \Xi A(t)(G(X(t)) - G(X(t))) - c(t) \eta \ddot{X}(t)^T \tilde{\Lambda} \ddot{X}(t)$$  \hspace{1cm} (24)
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Because,

\[ \ddot{X}(t)^T \Xi A(t) (G(X(t)) - G(X_c(t))) = -\frac{1}{N} \sum_{j=1}^{m} p_j \sum_{i>k} a_{ik} [x^i_j(t) - x^k_j(t)] [g_j(x^i_j(t)) - g_j(x^k_j(t))] \]

\[ \leq -\frac{1}{N} \sum_{j=1}^{m} p_j \sum_{i>k} a_{ik} [x^i_j(t) - x^k_j(t)]^2 \]

\[ = \frac{1}{N} \ddot{X}(t)^T (A(t) \otimes \bar{P}) \ddot{X}(t) \]

thus

\[ \dot{V}(t) \leq \ddot{X}(t)^T \Xi [F(X(t), t) - \Delta \ddot{X}(t)] + \ddot{X}(t)^T (U \Delta + \eta L \bar{\Delta}) \ddot{X}(t) \]

\[ + c(t) \ddot{X}(t)^T \left( \frac{1}{N} A(t) \otimes \bar{P} - \eta \bar{\Delta} \otimes \bar{\Gamma} \right) \ddot{X}(t) \]

The rest is just a repetition of the proof of Theorem 1, here we omit it. \( \square \)

Remark 5

In Theorem 2, condition \( \lambda_2(t) \leq \lambda^* < 0 \) plays a key role. However, calculating \( \lambda_2(t) \) for all time \( t \) is impossible numerically. But if all \( A(t) \) can be dominated by a constant matrix \( A^* \in \mathbb{A}_2 \), i.e. \( A(t) - A^* \in \mathbb{A}_2 \), then synchronization can be achieved under this new adaptive scheme.

3. NUMERICAL SIMULATIONS AND APPLICATIONS

In this section, we give several numerical simulations to demonstrate the effectiveness of the new proposed synchronization scheme with an adaptive coupling strength for coupled circuits systems.


In this simulation, we assume that the dynamics \( \dot{x}(t) = f(x(t)) \) of each uncoupled node is a neural network (see [20]) described by

\[ \dot{x}(t) = -D x(t) + C h(x(t)) + B h(x(t-1)) \quad (25) \]

where \( x(t) = (x^1(t), x^2(t))^T \in \mathbb{R}^2 \), \( h(x) = (\tanh(x^1), \tanh(x^2))^T \), and

\[ D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2.0 & -0.1 \\ -5.0 & 3.0 \end{bmatrix}, \quad B = \begin{bmatrix} -1.5 & -0.1 \\ -2.0 & -2.5 \end{bmatrix} \]

Its dynamical behavior with initial condition \( x^1(s) = 0.4, x^2(s) = 0.6, \forall s \in [-1, 0] \), is developed like chaotic attractors in Figure 1.

In addition in [20], the authors consider three linearly coupled such neural networks

\[ \dot{x}_i(t) = -D x_i(t) + C h(x_i(t)) + B h(x_i(t-1)) + c \sum_{j=1}^{3} a_{ij} x_j(t), \quad i = 1, 2, 3 \quad (26) \]
where the coupling matrix

\[ A = (a_{ij}) = \begin{pmatrix} -8 & 2 & 6 \\ 2 & -4 & 2 \\ 6 & 2 & -8 \end{pmatrix} \]

According to the discussion made in [20], to realize synchronization, the minimum theoretical value of \( c \) is 0.7529. On the other hand, it was also pointed out by simulations that for much smaller \( c \), identical synchronization can be realized already.

In the following, we apply the proposed adaptive scheme to find coupling strength \( c \), which is much smaller than the theoretical value, by the following algorithm

\[ \dot{x}_i(t) = -Dx_i(t) + Ch(x_i(t)) + Bh(x_i(t-1)) + c(t) \sum_{j=1}^{3} a_{ij}x_j(t), \quad i = 1, 2, 3 \]

\[ \dot{c}(t) = -\frac{\rho}{2} X(t)^T \tilde{A} X(t) \] (27)

where \( X(t) = (x_1(t)^T, x_2(t)^T, x_3(t)^T)^T, \ \tilde{A} = \tilde{A} \otimes I_2, \) and

\[ \tilde{A} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \] (28)
Pick the initial conditions as $x_1(s) = (0.4, 0.6)^T$, $x_2(s) = (0.3, 0.5)^T$, $x_3(s) = (0.1, 0.2)^T$, $s \in [-1, 0]$, and define $E(t) = \sqrt{\sum_{i,j=1}^{3} \|x_i(t) - x_j(t)\|^2}/3$ as a measure of the synchronization error. Let $\varepsilon = 0.001$, then the dynamics of the coupling strength $c(t)$ and the synchronization error $E(t)$ for coupled systems (27) are given in Figure 2. Obviously, in this case, $c(t) \rightarrow 0.098 \ll 0.7529$.

3.2. Simulation 2: Adaptive synchronization for linearly coupled Chua’s circuits

In this simulation, we assume that the dynamics $\dot{x}(t) = f(x(t))$ of each uncoupled node is described by the Chua’s circuit (see [41])

\begin{align*}
C_1 \frac{dV_1}{d\tau} &= \frac{1}{R}(V_2 - V_1) - h(V_1) \\
C_2 \frac{dV_2}{d\tau} &= \frac{1}{R}(V_1 - V_2) + i_L \\
L \frac{di_L}{d\tau} &= -V_2
\end{align*} \tag{29}

where $h(V_1) = G_b V_1 + \frac{1}{2}(G_a - G_b) \times [\|V_1 + E\| - \|V_1 - E\|]$ and its circuit diagram is described by Figure 3.

Therefore, via the re-scaling,

\begin{align*}
x^1 &= V_1/E, \quad x^2 = V_2/E, \quad x^3 = Ri_L/E \\
t &= \tau/RC_2, \quad a = RG_a, \quad b = RG_b \\
\varepsilon &= C_2/C_1, \quad \beta = C_2R^2/L
\end{align*} \tag{30}
Equation (29) is transformed into the simpler dimensionless form \( \dot{x}(t) = f(x(t)) \), where \( x(t) = (x_1(t), x_2(t), x_3(t))^T \):

\[
\begin{align*}
\frac{dx_1}{dt} &= \alpha(x_2 - x_1 - l(x_1)) \\
\frac{dx_2}{dt} &= x_1 - x_2 + x_3 \\
\frac{dx_3}{dt} &= -\beta x_2
\end{align*}
\]

where \( l(v) = bv + \frac{1}{2}(a - b)(|v + 1| - |v - 1|) \), for any \( v \in R \).

Picking the parameters \( \alpha = 9 \), \( \beta = \frac{100}{7} \), \( a = -\frac{2}{7} \), and \( b = -\frac{5}{7} \), the dynamical behavior of Equation (31) has a double-scroll chaotic attractor, as shown in Figure 4.
Now, we consider three linearly coupled Chua’s circuits

\[
\dot{x}_i(t) = f(x_i(t)) + c(t) \sum_{j=1}^{3} a_{ij} x_j(t), \quad i = 1, 2, 3
\]

\[
\dot{c}(t) = -\frac{2}{\alpha} X(t)^T \tilde{A} X(t)
\]

where \( X(t) = (x_1(t)^T, x_2(t)^T, x_3(t)^T)^T \), \( \tilde{A} = \tilde{A} \otimes I_3 \), and

\[
A = (a_{ij}) = \begin{pmatrix}
-3 & 2 & 1 \\
0 & -2 & 2 \\
1 & 1 & -2
\end{pmatrix}, \quad \tilde{A} = \begin{pmatrix}
-3 & 2 & 1 \\
2 & -4 & 2 \\
1 & 2 & -3
\end{pmatrix}
\]

Direct calculation indicates that the Jacobian matrix \( Df \) satisfies

\[
\frac{Df^T(x) + Df(x)}{2} \leq R = \begin{pmatrix}
1.2857 & 5 & 0 \\
5 & -1 & -6.6429 \\
0 & -6.6429 & 0
\end{pmatrix}
\]

whose eigenvalues are \(-8.6325, 0.8107, 8.1075\). Therefore, \( f \in \text{QUAD}(I_3, 10I_3, 0.6218) \).

Pick the initial value \( x_1(0) = (0.1, 0.2, 0.3)^T \), \( x_2(0) = (0.4, 0.5, 0.6)^T \), \( x_3(0) = (0.7, 0.8, 0.9)^T \), and define \( E(t) = \sqrt{\sum_{i,j=1}^{3} ||x_i(t) - x_j(t)||^2} / 3 \) as a measure of the synchronization error. Let \( \alpha = 0.0002 \), the dynamics of the coupling strength \( c(t) \) and the synchronization error \( E(t) \) for coupled systems (32) are given in Figure 5.

![Figure 5](image-url)
3.3. Simulation 3: Adaptive synchronization for coupled Lorenz systems

In this simulation, we choose the Lorenz system as the chaotic dynamical model of each uncoupled node: \( \dot{x}(t) = f(x(t)) \), which is described as

\[
\begin{align*}
\dot{x}^1 &= \sigma (x^2 - x^1) \\
\dot{x}^2 &= r x^1 - x^1 x^3 - x^2 \\
\dot{x}^3 &= x^1 x^2 - b x^3
\end{align*}
\]

where \( x = (x^1, x^2, x^3)^T \), \( \sigma = 10 \), \( r = 28 \), and \( b = \frac{8}{3} \), see Figure 6.

In [42], the authors proved rigorously that the coupled Lorenz system will be ultimately bounded and we can also synchronize them with coupling the second variable if the coupling is strong enough. Moreover, the following QUAD inequality has been proved.

**Proposition 1**

Suppose \( 0 < \omega < b \), \( x = (x^1, x^2, x^3)^T \), \( \hat{x} = (\hat{x}^1, \hat{x}^2, \hat{x}^3)^T \), and \( x^1(t)^2 + x^2(t)^2 + (x^3(t) - \sigma - r)^2 \leq B \), where \( B > 0 \) is a positive constant. Then,

\[
(x - \hat{x})^T P (f(x) - f(\hat{x}) - \Delta(x - \hat{x})) \leq -\omega (x - \hat{x})^T (x - \hat{x})
\]

(34)

holds, where \( P = \text{diag}\{\beta, 1, 1\} \) and \( \Delta = \text{diag}\{0, \delta, 0\} \), and

\[
\beta = \frac{B}{2\sigma(b - \omega)} + \frac{\omega}{\sigma}, \quad \delta = 2b - \sigma + \frac{(B - 2(b - \omega)(\sigma - \omega))^2}{2B(b - \omega)} - 1
\]
In the following, we simulate 100 linearly coupled Lorenz systems only via coupling the second variable

\[
\dot{X}(t) = F(X(t)) + c(t)AX(t)
\]

\[
\dot{c}(t) = -\frac{\gamma}{2}X^T(t)\tilde{A}X(t)
\]

(35)

where \(X(t) = (x_1^T(t), \ldots, x_{100}^T(t))^T\), \(F(X(t)) = (f(x_1(t))^T, \ldots, f(x_{100}(t))^T)^T\), \(A = A \otimes \Gamma, \tilde{A} = \tilde{A} \otimes \Gamma, \Gamma = \text{diag}(0, 1, 0)\).

For the coupling matrix \(A\), we first construct a coupling matrix of a small-world network generated by the method proposed in [4]; then replace every non-zero element \(a_{ij}, i \neq j\), with a positive random scalar, which is equally distributed in \([0, 1]\), and choose the diagonal elements to ensure \(A \in A_1\). So the matrix coupling is a weighted small-world network.

For convenience, we let \(\tilde{A} = (\tilde{a}_{ij}) \in A_2\) denote the globally connected matrix, i.e.

\[
\begin{align*}
\tilde{a}_{ii} &= -99, \quad i = 1, \ldots, 100 \\
\tilde{a}_{ij} &= 1, \quad i \neq j
\end{align*}
\]

(36)

According to Proposition 1 and Theorem 1, coupled systems (35) can be synchronized. We define

\[
E(t) = \sqrt{\frac{1}{100} \sum_{i,j=1}^{100} \|x_i(t) - x_j(t)\|^2 / 100}
\]

as a measure of the synchronization error. If we let \(\gamma = 1 \times 10^{-6}\), then the dynamics of the coupling strength \(c(t)\) and the synchronization error \(E(t)\) for coupled systems (35) are given in Figure 7.
3.4. Simulation 4: Adaptive synchronization for nonlinearly coupled Lorenz systems

Three nonlinearly coupled Lorenz systems with a time-varying coupling matrix and an adaptive coupling strength can be described as

\[
\dot{X}(t) = F(X(t)) + c(t)A(t)G(X(t)) \\
\dot{c}(t) = -\frac{\pi}{2} X^T(t)\tilde{A}X(t)
\]  

(37)

where \(X(t) = (x_1^T(t), x_2^T(t), x_3^T(t))^T\), \(F(X(t)) = (f(x_1(t))^T, f(x_2(t))^T, f(x_3(t))^T)^T\), and \(G(X(t)) = (g(x_1(t))^T, g(x_2(t))^T, g(x_3(t))^T)^T\), \(g(x_i(t)) = (0, x_i^2(t) + \tanh(x_i^2(t)), 0)^T\). \(A(t) = A(t)\otimes I_3, \tilde{A} = \tilde{A}\otimes I_3\), where \(\tilde{A}\) is also chosen as the globally coupled matrix as in Equation (33).

Here the time-varying coupling matrix \(A(t) \in \mathbb{A}_2\) is chosen as

\[
A(t) = \begin{pmatrix}
-5 - \sin t - \cos t & 3 + \sin t & 2 + \cos t \\
3 + \sin t & -5 - \sin t - \cos t & 2 + \cos t \\
2 + \cos t & 2 + \cos t & -4 - 2 \cos t
\end{pmatrix}
\]  

(38)

Obviously, it is easy to check that \(g(\cdot)\) satisfies the assumption in Theorem 2. Let

\[
A^* = \frac{1}{2} \begin{pmatrix}
-3 & 2 & 1 \\
2 & -3 & 1 \\
1 & 1 & -2
\end{pmatrix}
\]  

(39)

then \(A(t) - A^* \in \mathbb{A}_2\). So according to Remark 5, coupled systems (37) can be synchronized. We define

\[
E(t) = \sqrt{\sum_{i,j=1,2,3} \|x_i(t) - x_j(t)\|^2} / 3
\]
as a measure of the synchronization error. If we let $\alpha = 0.005$, then the dynamics of the coupling strength $c(t)$ and the synchronization error $E(t)$ for coupled systems (37) are given in Figure 8.

4. CONCLUSIONS

In this paper, we propose a new scheme to synchronize linearly or nonlinearly coupled systems with an adaptive coupling strength. Unlike those adaptive schemes existing in the literature, where coupled systems are synchronized to a special trajectory $s(t)$ or an equilibrium of the uncoupled system by adding a negative feedback controller, in this paper we synchronize the coupled systems with an adaptive coupling strength without knowing the synchronization trajectory, and this scheme can be applied to the cases that the coupling matrix is unknown or time-varying. Simulations are also given to demonstrate the effectiveness of the proposed scheme.

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