Cluster synchronization in networks of coupled nonidentical dynamical systems

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In this paper, we study cluster synchronization in networks of coupled nonidentical dynamical systems. The vertices in the same cluster have the same dynamics of uncoupled node system but the uncoupled node systems in different clusters are different. We present conditions guaranteeing cluster synchronization and investigate the relation between cluster synchronization and the unweighted graph topology. We indicate that two conditions play key roles for cluster synchronization: the common intercluster coupling condition and the intracluster communication. From the latter one, we interpret the two cluster synchronization schemes by whether the edges of communication paths lie in inter- or intracluster. By this way, we classify clusters according to whether the communications between pairs of vertices in the same cluster still hold if the set of edges inter- or intracluster edges is removed. Also, we propose adaptive feedback algorithms to adapting the weights of the underlying graph, which can synchronize any bi-directed networks satisfying the conditions of common intercluster coupling and intracluster communication. We also give several numerical examples to illustrate the theoretical results. © 2010 American Institute of Physics.

Cluster synchronization is considered to be more momentous than complete synchronization in brain science and engineering control, ecological science and communication engineering, and social science and distributed computation. Most of the existing works only focused on networks with either special topologies such as regular lattices or coupled two/three groups. For the general coupled dynamical systems, theoretical analysis to clarify the relationship between the (unweighted) graph topology and the cluster scheme, including both self-organization and driving, is absent. In this paper, we study this topic and find two essential conditions for an unweighted graph topology. We indicate that two conditions play key roles for cluster synchronization: the common intercluster coupling condition and the intracluster communication. Thus under these conditions, we present two manners of weighting to achieve cluster synchronization. One is adding positive weights on each edge with keeping the invariance of the cluster synchronization manifold and the other is an adaptive feedback weighting algorithm. We prove the availability of each manner. From these results, we give an interpretation of the two clustering synchronization schemes via the communication between pairs of individuals in the same cluster. Thus, we present one way to classify the clusters via whether the set of inter- or intracluster edges is removable if still wanting to keep the communication between vertices in the same cluster.

I. INTRODUCTION

Recent decades witness that chaos synchronization in complex networks has attracted increasing interests from many research and application fields since it was first introduced in Ref. 4. The word “synchronization” comes from Greek, which means “share time” and today, it comes to be considered as “time coherence of different processes.” Many new synchronization phenomena appear in a wide range of real systems, such as biology, neural networks, and physiological processes. Among them, the most interesting cases are complete synchronization, cluster synchronization, phase synchronization, imperfect synchronization, lag synchronization, almost synchronization, etc. See Ref. 8 and the references therein.

Complete synchronization is the most special one and characterized by that all oscillators approach to a uniform dynamical behavior. In this situation, powerful mathematical techniques from dynamical systems and graph theory can be utilized. Pecora et al. proposed the master stability function for transverse stability analysis of the diagonal synchronization manifold. This method has been widely used to study local complete synchronization in networks of coupled system. References 12–14 proposed a framework of Lyapunov function method to investigate global synchronization in complex networks. One of the most important issues is how the graph topology affects the synchronous
motion. As pointed out in Ref. 15, the connectivity of the graph plays a significant role for chaos synchronization. Cluster synchronization is considered to be more momentous in brain science and engineering control, and logical science and communication engineering, and social science and distributed computation. This phenomenon is observed when the oscillators in networks are divided into several groups, called clusters, by the way that all individuals in the same cluster reach complete synchronization but the motions in different clusters do not coincide.

Cluster synchronization of coupled identical systems is studied in Refs. 22–25. Among them, Jalan et al. pointed out two basic formations which realize cluster synchronization. One is self-organization, which leads to cluster synchronization via driving or and self-organizing configurations. Reference 23 proposed cluster synchronization scheme via dominant intracluster couplings and common intercluster couplings. Reference 26 studied local cluster synchronization for bipartite systems, where no intracluster couplings (driving scheme) exist. Reference 27 investigated global cluster synchronization in networks of two clusters with inter- and intracluster couplings. Belykh et al. studied this problem in one-dimensional and two-dimensional lattices of coupled identical dynamical systems in Ref. 22 and nonidentical dynamical systems in Ref. 28, where the oscillators are coupled via inter- or and intracluster manners. Reference 29 used nonlinear contraction theory to build up a sufficient condition for the stability of certain invariant subspace, which can be utilized to analyze cluster synchronization (concurrent synchronization is called in that literature). However, until now, there are no works revealing the relationship between the (unweighted) graph topology and the cluster scheme, including both self-organization and driving, for general coupled dynamical systems.

The purpose of this paper is to study cluster synchronization in networks of coupled nonidentical dynamical systems with various graph topologies. In Sec. II, we formulate this problem and study the existence of the cluster synchronization manifold. Then, we give one way to set positive weights on each edge and derive a criterion for cluster synchronization. This criterion implies that the communicability between each pair of individuals in the same cluster is essential for cluster synchronization. Thus, we interpret the two communication schemes according to the communication scheme among individuals in the same cluster. By this way, we classify clusters according to the manner by which synchronization in a cluster realizes. In Sec. III, we propose an adaptive feedback algorithm on weights of the graph to achieve a given clustering. In Sec. IV, we discuss the cluster synchronizability of a graph with respect to a given clustering and present the general results for cluster synchronization in networks with general positive weights. We conclude this paper in Sec. V.

II. CLUSTER SYNCHRONIZATION ANALYSIS

In this section, we study cluster synchronization in a network with weighted bidirected graph and a division of clusters. We impose the constraints on graph topology to guarantee the invariance of the corresponding cluster synchronization manifold and derive the conditions for this invariant manifold to be globally asymptotically stable by the Lyapunov function method. Before that, we should formulate the problem.

Throughout the paper, we denote a positive definite matrix $Z$ by $Z > 0$ and similarly for $Z < 0$, $Z \leq 0$, and $Z \geq 0$. We say that a matrix $Z$ is positive definite on a linear subspace $V$ if $u^T Z u > 0$ for all $u \in V$ and $u \neq 0$, denoted by $Z|_V > 0$. Similarly, we can define $Z|_V < 0$, $Z|_V \leq 0$, and $Z|_V \geq 0$. If a matrix $Z$ has all eigenvalues real, then we denote by $\lambda_h(Z)$ the $h$th largest eigenvalues of $Z$. $Z^T$ denotes the transpose of the matrix $Z$ and $Z^r=(Z+Z^T)/2$ denotes the symmetry part of a square matrix $Z$. #A denotes the number of the set $A$ with finite elements.

A. Model description and existence of invariant cluster synchronization manifold

A bidirected unweighted graph $G$ is denoted by a double set $\{V,E\}$, where $V$ is the vertex set numbered by $\{1, \ldots, m\}$, and $E$ denotes the edge set with $e(i,j) \in E$ if and only if there is an edge connecting vertices $j$ and $i$. $N(i)=\{j \in V : e(i,j) \in E\}$ denotes the neighborhood set of vertex $i$. The graph considered in this paper is always supposed to be simple (without self-loops and multiple edges) and bidirected. A clustering $C$ is a disjoint division of the vertex set $V \ni \{C_1, C_2, \ldots, C_K\}$ satisfying (i) $\bigcup_{k=1}^K C_k = V$; (ii) $C_k \cap C_l = \emptyset$ holds for $k \neq l$.

The network of coupled dynamical system is defined on the graph $G$. The individual uncoupled system on the vertex $i$ is denoted by an $n$-dimensional ordinary differential equation $\dot{\mathbf{x}}^i = f_i(\mathbf{x}^i)$ for all $i \in C_k$, where $\mathbf{x}^i = [x_1^i, \ldots, x_n^i]^T$ is the state variable vector on vertex $i$ and $f_i(\cdot): \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous vector-valued function. Each vertex in the same cluster has the same individual node dynamics. The interaction among vertices is denoted by linear diffusion terms. It should be emphasized that $f_k$ for different clusters are distinct, which can guarantee that the trajectories are apparently distinguishing when cluster synchronization is reached.

Consider the following model of networks of linearly coupled dynamical system: \[ \dot{\mathbf{x}}^i = f_k(\mathbf{x}^i) + \sum_{j \in N(i)} w_{ij} \Gamma(\mathbf{x}^i - \mathbf{x}^j), \quad i \in C_k, \quad k = 1, \ldots, K, \] where $w_{ij}$ is the coupling weight at the edge from vertex $j$ to $i$ and $\Gamma = [\gamma_{mn}]_{m,n=1}^p$ denotes the inner connection by the way that $\gamma_{mn} \neq 0$ if the $m$th component of the vertices can be 188 influenced by the $n$th component. The graph $G$ is bidirected and the weights are not requested to be symmetric.
Namely, we do not request $w_{ij} = w_{ji}$ for each pair $(i, j)$ with $e(i, j) \in \mathcal{E}$. Let $A = [a_{ij}]_{i,j=1}^{m}$ be the adjacent matrix of the graph $\mathcal{G}$. That is, $a_{ij} = 1$ if $e(i, j) \in \mathcal{E}$; otherwise, $a_{ij} = 0$. Then, model (1) can be rewritten as

$$\dot{x}^i = f_k(x^i) + \sum_{j=1}^{m} a_{ij}w_{ij}(x^j - x^i), \quad i \in \mathcal{C}_k, \quad k = 1, \ldots, K.$$  

In this paper, cluster synchronization is defined as follows. The differences among trajectories of vertices in the same cluster converge to zero as time goes to infinity, i.e.,

$$\lim_{t \to \infty} |x^i(t) - x^i(t)| = 0, \quad \forall i, \quad j \in \mathcal{C}_k, \quad k = 1, \ldots, K.$$  

The approach to analyze cluster synchronization is extended from that used in Ref. 14 to study complete synchronization. The general situation can be handled with the same approach and will be presented in Sec. IV. We denote the weighted Laplacian of the graph as $L = \sum_{ij} a_{ij}x_j - \sum_{i} a_{ii}x_i$. This demands $\alpha_{i,k'} = \alpha_{i,k''}$ for any $i \in \mathcal{C}_k$, $i' \in \mathcal{C}_k$, namely, $\alpha_{i,k'}$ is independent of $i$. Therefore, we have

$$\alpha_{i,k'} = \alpha(k,k'), \quad i \in \mathcal{C}_k, \quad k \neq k'.$$  

This condition is sufficient and necessary for the cluster synchronization manifold $S_{\mathcal{G}}(\alpha)$ to be invariant through the coupled system (2) for all $\alpha > 0$ and $\Gamma = I_m$. However, even though $\alpha > 0$ and $\Gamma = I_m$, the condition (8) also holds for sufficiently large $\alpha$ and $\Gamma = I_m$. In this paper, we suppose that the solution of the coupled system (2) is essentially bounded.

Throughout this paper, we assume that the vector-valued function $f_k(x) - \alpha \Gamma x: \mathbb{R}^n \to \mathbb{R}^n$ satisfies decreasing condition for some $\alpha \in \mathbb{R}$. That is, there exists $\delta > 0$ such that

$$\langle \xi - \zeta, f_k(\xi) - f_k(\zeta) - \alpha \Gamma (\xi - \zeta) \rangle \leq -\delta \langle \xi - \zeta, (\xi - \zeta) \rangle$$  

holds for all $\xi, \zeta \in \mathbb{R}^n$. This condition holds for any globally Lipschitz continuous function $f(\cdot)$ for sufficiently large $\alpha > 0$ and $\Gamma = I_m$. In this paper, we investigate cluster synchronization of networks of coupled nonidentical dynamical systems with the following weighting scheme:

$$w_{ij} = \begin{cases} c, & j \in \mathcal{N}_k(i) \quad \text{and} \quad \mathcal{N}_k(i) \neq \emptyset \\ d_{i,k}, & j \in \mathcal{N}_k(i) \quad \text{and} \quad \mathcal{N}_k(i) \neq \emptyset \\ 0, & \text{otherwise}, \end{cases}$$  

where $d_{i,k} = \#\mathcal{N}_k(i)$ denotes the number of elements in $\mathcal{N}_k(i)$ and $c$ denotes the coupling strength. Thus, the coupled system becomes

$$\dot{x} = \sum_{i=1}^{m} \sum_{j \in \mathcal{N}_k(i)} \frac{1}{N_k(i)} \Gamma(x^j - x^i).$$  

It can be seen that in Eq. (10), for each $i \in \mathcal{C}_k$, the corresponding $\alpha_{i,k'} = c$ for all $k' \in \mathcal{C}_k$ under the common intercluster coupling condition. The general situation can be handled with the same approach and will be presented in Sec. IV.

We denote the weighted Laplacian of the graph as $\mathcal{L}_k = \sum_{i \in \mathcal{C}_k} \sum_{j \neq i} a_{ij}(x_j - x_i)$. Thus, Eq. (10) can be rewritten as

$$\dot{x}_k = \sum_{j=1}^{m} l_{ij} \frac{x_j}{x_i}, \quad i \in \mathcal{C}_k, \quad k = 1, \ldots, K.$$  

The approach to analyze cluster synchronization is extended from that used in Ref. 14 to study complete synchronization. Let $d = [d_1, \ldots, d_m]^T$ be a vector with $d_i > 0$ for all $i = 1, \ldots, m$. We use the vector $d$ to construct a (skew) projection of $x = [x_1^T, \ldots, x_m^T]^T$ onto the cluster synchronization manifold $S_{\mathcal{G}}(\alpha)$. Define an average state with respect to $d$ in the cluster $\mathcal{C}_k$ as

$$\bar{x}_k = \frac{1}{\sum_{i \in \mathcal{C}_k} d_i} \sum_{i \in \mathcal{C}_k} d_i x_i.$$  

Thus, we denote the projection of $x$ on the cluster synchronization manifold $S_{\mathcal{G}}(\alpha)$ with respect to $d$ as $\bar{x}_k$.
Then, the variations $x^i - \bar{x}_d^i$ compose the transverse space
\[ \mathcal{T}_d^i(n) = \left\{ u = [u^1, \ldots, u^m]^T \in \mathbb{R}^m : \sum_{i \in \mathcal{C}_k} d_i u^i = 0, \quad \forall k \right\}. \]

In particular, in the case of $n = 1$, it denotes
\[ \mathcal{T}_d^i(1) = \left\{ u = [u^1, \ldots, u^m]^T \in \mathbb{R}^m : \sum_{i \in \mathcal{C}_k} d_i u^i = 0, \quad \forall k \right\}. \]

From the definition, we have the following lemma which is repeatedly used below.

**Lemma 1:** For each $k \in 1, \ldots, K$, it holds
\[ \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^i) = 0. \]

In fact, note
\[ \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^i) = \sum_{i \in \mathcal{C}_k} d_i x^i - \sum_{i \in \mathcal{C}_k} \left( \frac{1}{\sum_{j \in \mathcal{C}_k} d_j} \right) \sum_{i' \in \mathcal{C}_k} d_i x^{i'}. \]

The lemma immediately follows. As a direct consequence, we have
\[ \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^i)^T J_k = \left[ \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^i) \right]^T J_k = 0 \]
for any $J_k$ with a proper dimension independent of the index $i$.

Since the dimension of $\mathcal{T}_d^i(n)$ is $n(m - K)$, the dimension of $\mathcal{S}_d(n)$ is $nK$, and $\mathcal{S}_d(n)$ is disjoint with $\mathcal{T}_d^i(n)$ except the origin \[ \mathbb{R}^m = \mathcal{S}_d(n) \oplus \mathcal{T}_d^i(n), \] where $\oplus$ denotes the direct sum of linear subspaces. With these notations, the cluster synchronization manifold is equivalent to the transverse stability of the cluster synchronization manifold $\mathcal{S}_d(n)$, i.e., the projection of $x$ on the transverse space $\mathcal{T}_d^i(n)$ converges to zero as time goes to infinity.

**Theorem 1:** Suppose that the common intercluster coupling condition (7) holds, $\Gamma$ is symmetry and non-negative, and each vector-valued function $f_k(\cdot) - \alpha \Gamma \cdot$ satisfies the decreasing condition (8) for some $\alpha \in \mathbb{R}$. If there exists a positive definite diagonal matrix $D$ such that the restriction of $[D(cL + \alpha m)]^T$, restricted to the transverse space $\mathcal{T}_d^i(1)$, is nonpositive definite, i.e.,
\[ [D(cL + \alpha m)]^T |_{\mathcal{T}_d^i(1)} \leq 0 \]
holds, then the coupled system (11) can cluster synchronize with respect to the clustering $\mathcal{C}$.

**Proof:** We define an auxiliary function to measure the distance from $x$ to the cluster synchronization manifold as follows:
\[ V_k = \frac{1}{2} \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^i)^T (x^i - \bar{x}_d^i), \quad V(x) = \sum_{k=1}^{K} V_k. \]
Differentiating $V_k$ along Eq. (11) gives
\[ \dot{V}_k = \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^i)^T \left[ f_k(x^i) + c \sum_{j=1}^{m} l_{ij} \Gamma (x^j - \bar{x}_d^j) \right]. \]
Recalling the definitions of $l_{ij}$ and the common intercluster coupling condition (7), we have
\[ \sum_{j \in \mathcal{C}_{k'}} l_{ij} = \sum_{j \in \mathcal{C}_{k'}} l_{i'j'}, \quad \forall i, i' \in \mathcal{C}_k, \quad k \neq k', \]
which leads
\[ \sum_{j \in \mathcal{C}_{k'}} l_{ij} = \sum_{j \in \mathcal{C}_{k'}} l_{i'j'}, \quad \forall i, i' \in \mathcal{C}_k. \]
By Lemma 1, we have
\[ \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^i)^T = 0, \quad \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^i)^T f_k(x^i) = 0, \]
\[ \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^i)^T \left( \sum_{j \in \mathcal{C}_{k'}} l_{ij} \Gamma (x^j - \bar{x}_d^j) \right) = 0, \quad k' = 1, \ldots, K \]
due to the facts (13) and (14). Therefore, we have
\[ \dot{V}_k = \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^i)^T \left[ f_k(x^i) - f_k(x_d^i) + f_k(x_d^i) \right] + \sum_{k' = 1}^{K} \sum_{j \in \mathcal{C}_{k'}} l_{ij} \Gamma (x^j - \bar{x}_d^j) \]
\[ = \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^i)^T \left[ f_k(x^i) - f_k(x_d^i) \right] + \sum_{k' = 1}^{K} \sum_{j \in \mathcal{C}_{k'}} l_{ij} \Gamma (x^j - \bar{x}_d^j) \]
\[ \leq - \delta (w - v)^T (w - v), \]
we have
\[ \dot{V}_k \leq - \delta \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^i)^T (x^i - \bar{x}_d^i) + \sum_{i \in \mathcal{C}_k} d_i (x^i - \bar{x}_d^i)^T \]
\[ \times \left[ \sum_{k' = 1}^{K} \sum_{j \in \mathcal{C}_{k'}} l_{ij} \Gamma (x^j - \bar{x}_d^j) + \alpha \Gamma (x^i - \bar{x}_d^i) \right]. \]
Thus,
are also different, we are safe to say that the coupled system

\[ \dot{X} = -\delta \sum_{i=1}^{K} \sum_{j \in C_i} d_i(x_i - x_j)^T(x_i - x_j) + \sum_{i=1}^{K} d_i(x_i - x_j)^T \]

where \( \otimes \) denotes the Kronecker product and \( D = \text{diag}[d_1, \ldots, d_m] \).

Therefore,

\[ \dot{X} = (x - x_bar)^T \left[ \left( D + I_m \right) \otimes I_n \right] (x - x_bar) \]

(15)

Hence, we have

\[ \dot{V} \leq -\delta (x - x_bar)^T (D + I_n)(x - x_bar) = -2\delta \times V. \]

This implies that \( V(t) \leq \exp(-2\delta t) V(0) \). Therefore, \( \lim_{t \to \infty} V(t) = 0 \), namely, \( \lim_{t \to \infty} \sum_{i \in C_k} x_i(t) - x_bar(i) = 0 \) holds. In particular, \( \lim_{t \to \infty} x_i(t) - x_bar(i) = 0 \) for all \( i \in C_k \) and \( k = 1, \ldots, K \). According to the assumption that \( f_1(\cdot) \) are differentiable, if cluster synchronization is realized, the clusters are also different, we are safe to say that the coupled system (11) can cluster synchronize.

If each uncoupled system \( \dot{x} = f_i(x) \) is unstable, in particular, chaotic, \( \alpha \) must be positive in the inequality (8). It is natural to raise the question: Can we find some positive diagonal matrix \( D \) such that Eq. (12) satisfies with sufficiently large \( c \) and some certain \( \alpha \geq 0 \)? In other words, for the coupled system (10), what kind of unwighted graph topology \( G \) satisfying the common intercluster condition (7) can be a chaos cluster synchronizer with respect to the clustering \( C \). It can be seen that if the restriction of \( (DL + L^T D) \) to the transverse subspace \( T_0(1) \) is negative, i.e.,

\[ (DL + L^T D)|T_0(1)| < 0 \quad (16) \]

holds, then inequality (12) holds for sufficiently large \( c \).

With these observations, we have

**Theorem 2:** Suppose that the common intercluster coupling condition (7) holds for the coupled system (11) and \( \alpha > 0 \). There exist a positive diagonal matrix \( D \) and a sufficiently large constant \( c \) such that inequality (12) holds if and only if all vertices in the same cluster belong to the same connected component in the graph \( G \).

**Proof:** We prove the sufficiency for connected graph and unconnected graph separated.

**Case 1:** The graph \( G \) is connected. Then, \( L \) is irreducible. Perron–Frobenius theorem (see Ref. 32 for more details) tells us that the left eigenvector \( \{\xi_1, \ldots, \xi_n\}^T \) of \( L \) associated with the eigenvalue 0 has all components \( \xi_i > 0 \), \( i = 1, \ldots, m \). In this case, we pick \( d_i = \xi_i \), \( i = 1, \ldots, m \), and its symmetric part \( [DL]^T = (DL + L^T D)/2 \) has all row sums zero and irreducible with \( \lambda_1([DL]^T) = 0 \) associated with the eigenvector \( e = [1, \ldots, 1]^T \) and \( \lambda_2([DL]^T) < 0 \). Therefore, \( u^T(DL)u \leq \lambda_2([DL]^T)u^T u < 0 \) for any \( u \neq 0 \) satisfying \( u^Te = 0 \).

Now, for any \( u = [u_1, \ldots, u_m]^T \in \mathbb{R}^m \) with \( u^Du = 0 \), define \( \bar{u} = [\bar{u}_1, \ldots, \bar{u}_m]^T \), where \( \bar{u}_i = 1/m \sum_{j=1}^{m} u_{ij} \). It is clear that \( D \bar{u} = 0 \), \( \bar{u}^TL \bar{u} = 0 \), and \( (u - \bar{u})^Te = 0 \). Therefore,

\[ u^T(DL + L^T D)u = (u - \bar{u})^T(DL + L^T D)(u - \bar{u}) < 0, \]

since both hold. This implies that inequality (16) holds.

**Case 2:** The graph \( G \) is disconnected. In this case, we can divide the bigraph \( G \) into several connected components. If all vertices that belong to the same cluster are in the same connected component, then by the same discussion done in case 1, we conclude that inequality (16) holds for some positive definite diagonal matrix \( D \).

**Necessity:** We prove the necessity by reduction to absurdity. Considering an arbitrary disconnected graph \( G \), without loss of generality, supposing that \( L \) has form

\[ L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}, \]

and letting \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) correspond to the submatrices \( L_1 \) and \( L_2 \), respectively, we assume that there exists a cluster \( C_i \) satisfying \( C_i \cap \mathcal{V}_i = \emptyset \) for all \( i = 1, 2 \). That is, there exists at least a pair of vertices in the cluster \( C_i \) which cannot access each other. For each \( d = [d_1, \ldots, d_m]^T \) with \( d_i > 0 \) for all \( i = 1, \ldots, m \), letting \( D = \text{diag}[d_1, \ldots, d_m] \), we can find a nonzero vector \( e \in T_0(1) \) such that \( u^T(DL)u = 0 \) (see the Appendix for details). This implies that inequality (16) does not hold.

So, inequality (12) cannot hold for any positive \( \alpha \).

In the case that the clustering synchronized trajectories are chaotic with \( \alpha > 0 \), Theorem 2 tells us that chaos cluster synchronization can be achieved (for sufficiently large coupling strength) if and only if all vertices in the same cluster belong to the same connected component in graph \( G \).

In summary, the following two conditions play the key role in cluster synchronization:

1. common intercluster edges for each vertex in the same cluster and
2. communicability for each pair of vertices in the same cluster.

The first condition guarantees that the clustering synchronization manifold is invariant through the dynamical system with properly picked weights and the second guarantees that chaos clustering synchronization can be reached with a sufficiently large coupling strength.
TABLE I. Communicability of clusters under edge-removing operations.

<table>
<thead>
<tr>
<th>Cluster type</th>
<th>Remove the intracluster edges</th>
<th>Remove the intercluster edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cluster type A</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Cluster type B</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Cluster type C</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Cluster type D</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

C. Schemes to cluster synchronization

The theoretical results in Sec. II B indicate that the communica-
tion among vertices in the same cluster is important for chaos cluster synchronization. A cluster is said to be communicable if every vertex in this cluster can connect any other vertex by paths in the global graph. These possibilities of accessibility among all kinds of clusters in a connected graph. Moreover, it should be noticed that the cluster in the networks, as illustrated in Fig. 1, may not be connected via the subgraph topologies. For example, the first and third clusters in graph 1, the second and third clusters in graph 3, as well as all clusters in graph 2 are not connected by intercluster subgraph topologies. Certainly, the vertices in the same cluster are connected via inter- and intracluster edges. That is, we can realize cluster synchronization in non-clustered networks.

D. Examples

In this part, we propose several numerical examples to illustrate the theoretical results. In this example, we have $K=3$ clusters. The three graph topologies are shown in Fig. 1. The coupled system is

$$x^i = f_k(x^i) + c \left( \sum_{N_k(i) \neq i} \frac{1}{d_{i,k'}} \sum_{j \in N_k(i)} \Gamma(x^{i'} - x^{i}) \right),$$

where $\Gamma = \text{diag}[1,1,0]$ and $f_k(\cdot)$ are nonidentical Chua’s circuit

$$f_k(x) = \begin{cases} p_1[x_1 - x_2 + g(x_1)] \\ x_1 - x_2 + x_3 \\ -q_k x_2, \end{cases}$$

where $g(x_1) = m_2 x_1 + \frac{1}{2}(m_3 - m_2)(|x_1 + 1| - |x_1 - 1|)$. For all $k=1,2,3$, we take $m_2 = -0.68$ and $m_3 = -1.27$. The parameter pair $(\rho_k, q_k)$ distinguishes the clusters and is picked as $(10, 0.57)$, $(9, 1.07)$, $(9.0, 1.87)$, $(9.0, 1.287)$ for $k=1,2,3$, respectively.

As the Chua’s circuits are Lipschitz continuous, any $\alpha$ that is
TABLE II. Possibility of coexistence for two kinds of clusters in connected graph.

<table>
<thead>
<tr>
<th>Cluster type A</th>
<th>Cluster type B</th>
<th>Cluster type C</th>
<th>Cluster type D</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>√</td>
<td>X</td>
<td>X</td>
</tr>
<tr>
<td>Cluster type B</td>
<td></td>
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<td>X</td>
</tr>
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<td>X</td>
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<tr>
<td>Cluster type C</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Cluster type D</td>
<td>X</td>
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</tr>
</tbody>
</table>

510 greater than the maximum of the Lipschitz constant of $f_k$ can 511 satisfy the decreasing condition. We use the following quantity 512 to measure the variation for vertices in the same cluster:

$$\text{var} = \left( \sum_{k=1}^{K} \frac{1}{\#C_k - 1} \sum_{i \in C_k} (x_i - \bar{x}_k)^T (x_i - \bar{x}_k) \right)^{1/2},$$

514 where $\bar{x}_k = 1/\#C_k \sum_{i \in C_k} x_i$, $\langle \cdot \rangle$ denotes the time average. The 515 ordinary differential equations (17) are solved by the Runge– 516 Kutta fourth-order formula with a step length of 0.001–0.01 517 according to the size of the coupling strength. The time interval for computing the average is [50, 100].

519 Figure 2 indicates that for either graph 1, graph 2, or graph 3, the 520 coupled system (17) clustering synchronizes, respectively, if 521 the coupling strength is larger than certain threshold value. 522 The threshold for each graph observed by the plots is clearly 523 larger than the theoretical results, which will be shown in 524 details in Sec. IV A. It is not surprising since the theoretical 525 results only give a sufficient condition that the coupled system 526 can cluster synchronize if the coupling strength $c$ is large 527 enough. It does not exclude the case that the coupled system 528 can still cluster synchronize even if the coupling strength $c$ is 529 small.

The following quantity is used to measure the deviation 531 between clusters:

$$\text{dis}(t) = \min_{i \neq j} [\bar{x}_i(t) - \bar{x}_j(t)]^T [\bar{x}_i(t) - \bar{x}_j(t)].$$

532 Figure 3 shows that the deviation between clusters is apparent, even var = 0, where the coupling strengths are picked in 533 the theoretical region guaranteeing clustering synchronization. 534 It is clear that the difference is caused by the different 535 choice of parameters for different clusters. This illustrates that 536 the cluster synchronization is actually realized.

FIG. 3. (Color online) Dynamics of dis(t) through Eq. (10): (a) for graph 1 with $c=20$; (b) for graph 2 with $c=53$; (c) for graph 3 with $c=53$, respectively.

III. ADAPTIVE FEEDBACK CLUSTER SYNCHRONIZATION ALGORITHM

539 For a certain network topology, which has weak cluster synchronizability, i.e., the threshold to ensure clustering synchronization is relatively large, which is further studied in Sec. IV A. It is natural to raise the following question: 544 How to achieve cluster synchronization for networks no mat-
ter they have “good” topology or not. One approach pro-
posed recently is adding weights to vertices and edges. Ref-
erece 35 showed evidences that certain weighting
procedures can actually enhance complete synchronizability.
On the other hand, adaptive algorithm has emerged as an
efficient means of weighting to actually enhance complete
synchronizability.36

In this section, we consider the coupled system

\[ x^i = f_k(x^i) + \sum_{j=1}^{m} a_{ij}w_{ij} \Gamma(x^j - x^i), \quad i \in C_k, \quad k = 1, \ldots, K \]  

(19)

\[ \dot{x}(t) = f_k(x(t)) + \sum_{j=1}^{m} a_{ij}w_{ij}(t) \Gamma[x(t) - x^i(t)], \quad i \in C_k, \quad k = 1, \ldots, K \]

(20)

\[ \dot{w}_{ij}(t) = \rho_{ij}d_t[x(t) - x^i(t)] \Gamma[x(t) - x^i(t)] \]

for each \( e_{ij} \in E \) and \( i \in C_k, \quad k = 1, \ldots, K \)

with \( \rho_{ij} > 0 \) as constants.

**Theorem 3:** Suppose that graph \( G \) is connected, all the
assumptions of Theorem 1 hold, the system (20) is essentially
bounded. Then the system (20) cluster synchronizes for any
initial data.

**Proof:** First of all, pick \( l_j \) as defined in Eq. (11) and a
sufficiently large \( c \). Since \( G \) is connected, Theorem 2 tells

\[ [D(cL + ad_m)]^T \rho_{ij} < 0. \]  

(21)

Define the following candidate Lyapunov function

\[ Q_k(x, W) = \sum_{i \in C_k} \left[ \frac{d_i}{2}(x^i - x^k_d)^T(x^i - x^k_d) + \frac{1}{2}a_{ij}(w_{ij} - cl_{ij})^2 \right] \]

\[ Q(x, W) = \sum_{k=1}^{K} Q_k. \]

Differentiating \( Q_k \), we have

\[ \dot{Q}_k = \sum_{i \in C_k} d_i(x^i - x^k_d)^T \left\{ f_k(x^i) + \sum_{j=1}^{m} a_{ij}w_{ij} \Gamma(x^j - x^i) \right\} \]

+ \[ \sum_{i \in C_k} \sum_{j=1}^{K} \sum_{l=1}^{N_{ij}} \sum_{j=1}^{N_{ij}} a_{ij}(w_{ij} - cl_{ij})d_i(x^j - x^k_d)^T \Gamma(x^j - x^i) \]

\[ = \sum_{i \in C_k} d_i(x^i - x^k_d)^T \left\{ f_k(x^i) + c \sum_{j=1}^{m} l_{ij} \Gamma(x^j - x^i) - x^k_d \right\} \].

Similar to the proof of Theorem 1, we have

\[ \sum_{i \in C_k} \dot{Q}_k = \sum_{i \in C_k} d_i(x^i - x^k_d)^T \]

\[ \times \left\{ f_k(x^i) - f(x^s) + c \sum_{j=1}^{m} l_{ij} \Gamma(x^j - x^s_d) \right\} \]

and

\[ \dot{Q} = \sum_{k=1}^{K} \dot{Q}_k \leq -\delta \sum_{k=1}^{K} \sum_{i \in C_k} d_i(x^i - x^k_d)^T(x^i - x^k_d) \]

\[ + \sum_{k=1}^{K} \sum_{i \in C_k} d_i(x^i - x^k_d)^T \]

\[ \times \left\{ a_{ij}(x^j - x^s_d) + c \sum_{j=1}^{m} l_{ij} \Gamma(x^j - x^s_d) \right\} \]

\[ = -\delta(x - x_s)^T(D \otimes I)(x - x_s) + (x - x_d)^T \]

\[ \times \left\{ [D(cL + ad_m)]^T \Gamma \right\}(x - x_d). \]

Inequality (21) implies

\[ \dot{Q} \leq -\delta(x - x_d)^T(D \otimes I)(x - x_d) \leq 0. \]

This implies

\[ \int_{0}^{t} \delta(x(s) - x_d(s))^T(D \otimes I)(x(s) - x_d(s))ds \leq Q(0) \]

\[ -Q(t) \leq Q(0) < \infty. \]  

(22)

From the assumption of the boundedness of Eq. (20), we can conclude \( \lim_{t \to \infty} [x(t) - x_d(t)] = 0 \) due to the fact that \( x(t) \) is uniform continuous. This completes the proof.

For the disconnected situation, we can split the graph into several connected components and deal with each con-
nected component by the same means as above. The dynam-
ics of the weights $w_{ij}(t)$ is an interesting issue. Even though it is illustrated in Fig. 4 that all weights converge, to our best reasoning, we can only prove that all intraweights converge, i.e., vertices $i$ and $j$ belonging to the same cluster $C_k$. In fact, by Eq. (22), we have

$$\int_0^\infty |\dot{w}_{ij}(\tau)|d\tau = \rho_{ij}d_2 \int_0^\infty \left[ |x_i'(\tau) - \bar{x}_i(\tau)| \right]_2 \Gamma [x_i'(\tau) - x_i'(\tau)]d\tau$$

$$\leq \int_0^\infty \rho_{ij}d_2 \left\{ \left[ |x_i'(\tau) - \bar{x}_i(\tau)| \right]_2 \Gamma [x_i'(\tau) - x_i'(\tau)] + \left[ |x_i'(\tau) - \bar{x}_i(\tau)| \right]_2 \Gamma [x_i'(\tau) - x_i'(\tau)] \right\}d\tau$$

$$\leq \rho_{ij}d_2 \left\{ \frac{3}{2} \int_0^\infty \left[ |x_i'(\tau) - \bar{x}_i(\tau)| \Gamma [x_i'(\tau) - x_i'(\tau)]d\tau + \frac{1}{2} \int_0^\infty \left[ |x_i'(\tau) - \bar{x}_i(\tau)| \Gamma [x_i'(\tau) - x_i'(\tau)]d\tau \right\} .

Thus,

$$f_k(u) = \begin{cases} 10(u_2 - u_1) & \\
\frac{u_1}{3}u_2 - u_2 - u_1u_3 & \\
u_1u_3 - b_2u_3 \end{cases}$$

(23)

where the parameters $b_1=28$ for the first cluster, $b_2=38$ for the second cluster, and $b_3=58$ for the third cluster are used to distinguish the clusters. As shown in Ref. 37, the boundedness of the trajectories of an array of coupled Lorenz systems can be ensured. Also, this bound is independent of the coupling strength. Therefore, the decreasing condition (8) can be satisfied for a sufficiently large $\alpha$. In fact, the theoretical estimation of such $\alpha$ is rather large and much larger than the simulating observation (not shown in this paper). However, Theorem 3 indicates that the existence of such $\alpha$ (even very large in theory) is sufficient for the adaptive feedback algorithm (20) succeeding in clustering synchronizing the coupled system.

The ordinary differential equations are solved by the Runge–Kutta fourth-order formula with a step length 0.005. The initial values of the states and the weights are randomly picked in $[-3, 3]$ and $[-5, 5]$, respectively. We use the following quantity to measure the state variance inside each cluster with respect to time:

$$K(t) = \sum_{k=1}^\kappa \frac{1}{\#C_k - 1} \sum_{i \in C_k} [x_i(t) - \bar{x}_i(t)]^T [x_i(t) - \bar{x}_i(t)] .

Figure 6 shows that the adaptive algorithm succeeds in clustering synchronizing the network with respect to the pre-given clusters. Figure 7 indicates that the differences between clusters are due to nonidentical parameters $b_\ell$. As shown in Fig. 4, the weights converge but the limit values are not always positive. This is not surprising. The right-hand side of the algorithm (20) can be either positive or negative, which causes some weights of edges to be negative. Discussion of the situation with negative weights is out of the scope of this paper.

A. Examples

To illustrate the adaptive feedback algorithms, we still use graphs 1–3 described in Fig. 1 as the network topology. Also this time we use the Lorenz system as the uncoupled system,

$$\int_0^\infty [x_i'(\tau) - \bar{x}_i(\tau)]^T [x_i'(\tau) - \bar{x}_i(\tau)]d\tau < + \infty .

Thus,

$$f_k(u) = \begin{cases} 10(u_2 - u_1) & \\
\frac{u_1}{3}u_2 - u_2 - u_1u_3 & \\
u_1u_3 - b_2u_3 \end{cases}

(23)

where the parameters $b_1=28$ for the first cluster, $b_2=38$ for the second cluster, and $b_3=58$ for the third cluster are used to distinguish the clusters. As shown in Ref. 37, the boundedness of the trajectories of an array of coupled Lorenz systems can be ensured. Also, this bound is independent of the coupling strength. Therefore, the decreasing condition (8) can be satisfied for a sufficiently large $\alpha$. In fact, the theoretical estimation of such $\alpha$ is rather large and much larger than the simulating observation (not shown in this paper). However, Theorem 3 indicates that the existence of such $\alpha$ (even very large in theory) is sufficient for the adaptive feedback algorithm (20) succeeding in clustering synchronizing the coupled system.

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$$K(t) = \sum_{k=1}^\kappa \frac{1}{\#C_k - 1} \sum_{i \in C_k} [x_i(t) - \bar{x}_i(t)]^T [x_i(t) - \bar{x}_i(t)] .

Figure 6 shows that the adaptive algorithm succeeds in clustering synchronizing the network with respect to the pre-
IV. DISCUSSIONS

In this section, we make further discussions for some closely relating issues.

A. Clustering synchronizability

Synchronizability is used to measure the capability of synchronzation for the graph. It can be described by the threshold of the coupling strength to guarantee that the coupled system can synchronize. For complete synchronizability, it was formulated as a function of the eigenvalues of symmetric Laplacian or certain Rayleigh quotient of asymmetry, it was formulated as a function of the eigenvalues of coupled system. Theorem 1 tells us that under the common intercluster condition, the synchronization for the graph. It can be described by the study of complex networks. Here, similarly, we are also interested in how to formulate and analyze the cluster synchronizability of a graph $G$ and a clustering $C$.

Consider the model (11) of coupled system. Theorem 1 tells us that under the common intercluster condition, the cluster synchronization condition (12) can be rewritten as

$$c > \frac{\alpha}{\min_{u \in \mathcal{D}_u, u \neq 0} -u^\top(DL)u}$$

for some positive definite diagonal $D$. Therefore, we take the Rayleigh–Hitz quotient

$$\text{CS}_{G,C} = \max_{D \in \mathcal{D}} \min_{u \in \mathcal{D}_u, u \neq 0} -u^\top(DL)u$$

to measure the cluster synchronizability for graph $G$ and clustering $C$, where $\mathcal{D}$ denotes the set of positive definite diagonal matrices of dimension $m$. It can be seen that the larger the $\text{CS}_{G,C}$ is, the smaller the coupling strength $c$ can be, such that the coupled system (11) clusteringly synchronizes.

In particular, if $L$ is symmetric, then $\text{CS}_{G,C}$ is just the maximum eigenvalue of $-L$ in the transverse space $T_c(1)$, where $c=[1,1,\ldots,1]^\top$. It is an interesting topic about how the two schemes discussed above affect the cluster synchronizability for a given graph topology. It will be a possible topic in our future research.

Reconsidering the examples in Sec. II D, we can use MATLAB LMI and Control Toolbox to obtain the numerical values of $\text{CS}_{G,C}$ for three graphs shown in Fig. 1. Thus, we can derive the values of $\text{CS}_{G,C}$: 0.472, 0.178, and 0.172, respectively. So, we can obtain the minimal estimation of the coupling strength $c$ as

$$c^* = \frac{\alpha}{\text{CS}_{G,C}}.$$odels.

The globally Lipschitz continuity of Chua’s circuit allows us to obtain $\alpha < 9.062$. Thus, we obtain estimations of the infimum of $c$: 19.20 for graph 1, 50.91 for graph 2, and 52.69 for graph 3. The details of algebras are omitted here. One can see that they are all located in the region of cluster synchronizability, as numerically illustrated in Fig. 2, but less accurate since the estimation of $\alpha$ is very loose. However, the theoretical value of $\text{CS}_{G,C}$ provides information on the relative
synchronizability of coupling structure, independent of the
node dynamics set on the network.

32 B. Generalized weighted topologies

Previous discussions can also be available toward the
coupled system (2) with general weights,

\[ x' = f_k(x') + \sum_{j=1}^{m} a_{ij}w_{ij}(x' - x'), \quad i \in C_k, \quad k = 1, \ldots, K. \]  

35

36 Here, the graph may be directed, i.e., \( a_{ij} = 1 \), if there is an
dge from vertex \( j \) to vertex \( i \), otherwise, \( a_{ij} = 0 \). Weights are
even not required positive. For the existence of invariant
cluster synchronization manifold, we assume

\[ \sum_{j \in N_k(i)} w_{ij} = \sum_{j' \in N_k(i')} w_{i'j'}, \quad 1 \leq i, i' \leq K. \]  

38

39 holds for all \( i, i' \in C_k \) and \( k \neq k' \). Define its Laplacian
34 G=[g_{ij}]_{j=1}^{m} as follows:

\[ g_{ij} = \begin{cases}  
  w_{ij}, & a_{ij} = 1 \\
  0, & i \neq j \text{ and } a_{ij} = 0 \\
  -\sum_{k=1, k \neq i}^{m} g_{ik}, & i = j.
\end{cases} \]  

42

43 Thus, Eq. (25) becomes

\[ x' = f_k(x') + \sum_{j=1}^{m} g_{ij} \Gamma(x') - x', \quad i \in C_k, \quad k = 1, \ldots, K. \]  

46

47 Replacing \( c_{ij} \) by \( g_{ij} \) and following the routine of the
48 proof of Theorem 1, we can obtain following results.

49 Theorem 4: Suppose that the common intercluster cou-
50 ping condition (26) is satisfied, each \( f_k(\cdot) - \alpha \Gamma \) satisfies the
51 decreasing condition for some \( \alpha \in \mathbb{R} \), and \( \Gamma \) is non-negative
52 definite. If there exists a positive definite diagonal matrix \( D \)
53 such that

\[ \left[ D(G + \alpha D_{m}) \right]_{\mathbb{R}^{n \times n}} \preceq 0 \]  

56

57 holds, then the coupled system (27) can cluster synchronize
58 with respect to the clustering \( C \).

59 Also, we use the same discussions as in Theorem 2 to
60 obtain the following general result.

61 Theorem 5: Suppose that the common intercluster cou-
62 ping condition (7) is satisfied. For a bidirected unweighted
63 graph \( G \), there exist positive weights to the graph \( G \) such that
64 inequality (28) holds if and only if all vertices in the same
65 cluster belong to the same connected component in graph \( G \).

68 In fact, the proofs of Theorems 4 and 5 simply repeat
69 those of Theorems 1 and 2, respectively.

70 Here, we compare the results in a closely relating work, and
71 with this paper. First, investigate the local cluster synchroni-
72 zation of interconnected clusters by extending the master sta-
73 bility function method. Instead, in this paper, we are con-
74 cerned with the global cluster synchronization. Second, the
75 models of the two papers are different. The topologies dis-
76 cussed in Ref. 26 exclude intracluster couplings. In this pa-
77 per, we consider more general graph topology. Third, Ref. 26
78 studied the situation of nonlinear coupling function and we
79 consider the linear case. Despite that Ref. 26 considered dif-
80 ferent coupling strengths for clusters and we consider a com-
81 mon one in Sec. II, Theorem 4 can apply to discussion of the
82 models proposed in Ref. 26, too.

V. CONCLUSIONS

The idea for studying synchronization in networks of
coupled dynamical systems sheds light on cluster synchroni-
ization analysis. In this paper, we study cluster synchroniza-
tion in networks of coupled nonidentical dynamical systems.
Cluster synchronization manifold is defined as that the dy-
namics of the vertices in the same cluster are identical. The
varity criterion for cluster synchronization is derived via linear ma-
trix inequality. The differences between clustered dynamics
are guaranteed by the nonidentical dynamical behaviors of
767 different clusters. The algebraic graph theory tells that the
878 communicability between each pair of vertices in the same
879 cluster is a doorsill for chaos cluster synchronization. This
399 leads to a description of two schemes to realize cluster syn-
chronization: the set of intracluster edges is irremovable for
422 the communication between each pair of vertices in the same
453 cluster; the set of intercluster edges is irremovable for the
484 communication between vertices in the same cluster. One
404 can see that the latter scheme implies that cluster synchroni-
435 zation can be realized in a network without community struc-
466 ture, for example, graph 2 in Fig. 1. Adaptive feedback al-
497 gorithm is used to enhance cluster synchronization.

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APPENDIX: PROOF OF NECESSITY IN THEOREM 2

In this appendix, for each positive \( d \), we give the details
806 to find a \( u \in T_{\mathbb{R}^{n \times n}} \) with \( u \neq 0 \), such that \( u^{\top} D L u = 0 \) in the
831 case that there exists a cluster \( C_1 \) that does not belong to the
862 same connected component. Without loss of generality, sup-
893 pose \( L \) has the following form:

\[ L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}. \]  

815

Let \( V_1 \) and \( V_2 \) correspond to submatrices \( L_1 \) and \( L_2 \), respec-
846 tively, and \( C_1 \cap \nabla_i \neq \emptyset \) for all \( i = 1, 2 \). There are two cases.
877 First, in the case that \( C_1 \) is isolated from other clusters.
In this
810 case, there are no edges connecting \( C_1 \) to other clusters. 819
820 Define

\[ L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}. \]  

815

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846 tively, and \( C_1 \cap \nabla_i \neq \emptyset \) for all \( i = 1, 2 \). There are two cases.
877 First, in the case that \( C_1 \) is isolated from other clusters. 819
820 Define
In the second case, $C_1$ is not isolated. Suppose the net-work has $K$ clusters and $L_1$ and $L_2$ are connected (otherwise, we only consider the connection components of $L_1$ and $L_2$ that contain vertices from $C_1$). Due to the common intercluster coupling condition and the absence of isolated cluster, we have $C_i \cap V_j \neq \emptyset$ for all $i = 1, \ldots, K$ and $j = 1, 2$. Pick a vector $u = [u_1, \ldots, u_K]$. Letting $u_i = \begin{cases} \alpha_i, & i \in C_1 \cap V_1 \\ \beta_i, & i \in C_2 \cap V_2 \\ 0, & \text{otherwise,} \end{cases}$

$$u_i = \begin{cases} \alpha_i, & i \in C_1 \cap V_1 \\ \beta_i, & i \in C_2 \cap V_2 \\ 0, & \text{otherwise,} \end{cases}$$

821

$$u_i = \begin{cases} \alpha_i, & i \in C_1 \cap V_1 \\ \beta_i, & i \in C_2 \cap V_2 \\ 0, & \text{otherwise,} \end{cases}$$

822

$$u = \sum_{j \in C_1 \cap V_1} d_j, \quad b = \sum_{j \in C_2 \cap V_2} d_j.$$ Then, by picking $\alpha$ and $\beta$ satisfying $\alpha + \beta = 0$, we have $u \in T_k^1(1)$. In addition, $u^T D_L u = 0$ due to $L_L = 0$.

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